A Note on α/β-Shapley NTU value in strategic form games

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Abstract

In this note, we show that in a two player zero-sum game which has a Nash equilibrium, the α/β Shapley NTU value is equivalent to the minmax value of the game and prove the non-emptiness of α/β Shapley NTU value.

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Key Words: Shapley value, NTU game, strategic form game

1 Introduction

In this note, we extend the notion of α/β to *Shapley NTU(non-transferable utility) value*, which is also known as λ*-transfer NTU value* or *Shapley solution*, and prove the non-emptiness of α/β Shapley NTU value, which is Shapley value of NTU α and β characteristic function defined over a strategic form game.

The α/β theory was first introduced in Aumann and Peleg (1960). In the core theory, this concept is well developed as α/β core , for example, Scarf (1971), Zhao(1999a, 1999b), etc. However, in the value theory, the α/β theory is not explicitly applied yet. One of the well known value is the Shapley value, which was firstly defined and axiomized for TU (transferable utility) games in Shapley (1953), and which was later extended to NTU games in Shapley (1969), where he proved the existence of Shapley NTU value. Many excellent studies of Shapley NTU value are provided by many contributers , e.g., Shafer (1980), Aumann (1985), etc. We will focus on the linkage of Shapley NTU value with strategic form games.

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2 Definitions

Let R^n be an n-dimensional Euclidean space. Let $G = \{N, (u_i)_{i\in N}, (X^i)_{i\in N}\}$ be a strategic form game, where N is a finite set such that $|N| = n$ [1] , for each $i \in N$, X^i is a nonempty, compact, and convex subset of R^n, and for each $i \in N$, u^i is a real valued continuous (utility) function such that $u_i : X \to R$, where $X \stackrel{\text{def}}{=} \prod_{i\in N} X^i$. Let $\mathcal{N}$ be a set of non-empty coalition of N, i.e., $\mathcal{N} \stackrel{\text{def}}{=} 2^N \setminus \{\emptyset\}$. Define $X^S \stackrel{\text{def}}{=} \prod_{i\in S} X^i$ for $S \in \mathcal{N}$. Define the α-characteristic function as $V_\alpha(S) \stackrel{\text{def}}{=} \bigcup_{x^S\in X^S} \bigcap_{x^{N\setminus S}\in X^{N\setminus S}} \{u \in R^N | \forall i \in S, u_i \leq u_i(x^S, x^{N\setminus S})\}$, and define the β-characteristic function as $V_\beta(S) \stackrel{\text{def}}{=} \bigcap_{x^{N\setminus S}\in X^{N\setminus S}} \bigcup_{x^S\in X^S} \{u \in R^N | \forall i \in S, u_i \leq u_i(x^S, x^{N\setminus S})\}$.

Let V be an NTU game such that $V : \mathcal{N} \to R^N$. An NTU game V is called *super-additive*, if $V(S) \cap V(T) \subset V(S \cup T)$ for all S,$T \in \mathcal{N}$ such that $S \cap T = \emptyset$. An NTU game V is called *comprehensive*, if $u \in V(S)$ then $u' \in V(S)$ for all $u' \leq u$, for all $S \in \mathcal{N}$.

3 Results

The next lemma is a slightly modified version of Aumann(1961, Section 9).

Lemma 1. *For a strategic form game G such that u_i is quasi-concave in X for all $i \in N$, the α/β characteristic functions V_α and V_β are (1) non-empty, (2) closed valued, (3) bounded above, (4) comprehensive, and (5) super-additive. In addition, if u_i is concave in X for all $i \in N$, then (6) V_α is convex valued and (6') $V_\beta(N)$ is a convex set.*

Proof. (1),(2),(3), and (4) are obvious by definition and the continuity of utility function. (5) of V_α follows from the following fact. If $u \in V_\alpha(S)$ and $u \in V_\alpha(T)$ for disjoint coalition $S, T \in \mathcal{N}$, then $\exists \bar{x}^S$, $\forall (x^T, x^{N\setminus(S\cup T)})$, $\forall i \in S$, $u_i \leq u_i(\bar{x}^S, x^T, x^{N\setminus(S\cup T)})$ and $\exists \bar{x}^T$, $\forall (x^S, x^{N\setminus(S\cup T)})$, $\forall j \in T$, $u_j \leq u_j(\bar{x}^T, x^S, x^{N\setminus(S\cup T)})$. Hence $\exists \bar{x}^S$ and $\bar{x}^T$, $\forall x^{N\setminus(S\cup T)}$, $\forall i \in S \cup T$, $u_i \leq u_i(\bar{x}^S, \bar{x}^T, x^{N\setminus(S\cup T)})$.

Let us prove (5) of V_β. Set $\bar{u} \in V_\beta(S) \cap V_\beta(T)$ and $x^{N\setminus(S\cup T)} \in X^{N\setminus(S\cup T)}$. Construct a set valued mapping $F : X^S \times X^T \twoheadrightarrow X^S \times X^T$ such that $F(x^S, x^T) = C(x^T) \times D(x^S)$, where $C(x^T) = \{x^S \in X^S | \forall i \in T, u_i(x^S, x^T, x^{N\setminus(S\cup T)}) \geq \bar{u}^T\}$ and $D(x^S) = \{x^T \in X^T | \forall j \in S, u_j(x^S, x^T, x^{N\setminus(S\cup T)}) \geq \bar{u}^S\}$, which is well defined. If the mapping F has a fixed point $(x^{S^*}, x^{T^*}) \in F(x^{S^*}, x^{T^*})$, then $u_i(x^{S^*}, x^{T^*}, x^{N\setminus(S\cup T)}) \geq \bar{u}_i$ for all $i \in S \cup T$, for each $x^{N\setminus(S\cup T)} \in X^{N\setminus(S\cup T)}$. It means that $\bar{u} \in V_\beta(S \cup T)$. The mapping F is convex valued because of the quasi-concavity of u_i. The closed valuedness follows from the continuity of u_i. Thus F is compact valued. We can also easily check the upper-hemicontinuity of F. Applying Kakutani's fixed point

[1] $|N|$ means the cardinality of N.

theorem, we obtain the existence of a fixed point (x^{S^*}, x^{T^*}) of F. (6) and (6') follow from the concavity of u_i. *Q.E.D.*

Note that the super-additivity of V_β may require additional conditions, i.e., the continuity and quasi-concavity of utility functions while the super-additivity of V_α is obtained by definition. Even if u_i is concave, V_β is not convex valued in general. However, the existence of Shapley NTU value does not need the convex valuedness of characteristic functions though the convexity of $V(N)$ is required.

Let v be a TU game such that $v : \mathcal{N} \to R$ and $v(\emptyset) = 0$. The Shapely (TU) value $\varphi(v) = (\varphi_1(v), \cdots, \varphi_n(v))$ is defined by the condition

$$\varphi_i(v) = \sum_{S \subset N} (v(S) - v(S \setminus \{i\})) \frac{(|S|-1)!(|N|-|S|)!}{|N|!}$$

for each $i \in N$. Set $\Delta^{n-1} \stackrel{\text{def}}{=} \{\lambda \in R^N_+ | \sum_{i \in N} \lambda_i = 1\}$, and for a NTU game V, for each $\lambda \in \Delta^{n-1}$, define a TU game $v_\lambda : \mathcal{N} \to R$ by $v_\lambda(S) \stackrel{\text{def}}{=} \sup\{\sum_{i \in S} \lambda_i u_i | u \in V(S)\}$. If $V = V_\alpha$ (V_β), then denote v_λ as v^α_λ (v^β_λ resp.)

Definition 1 (α (β) Shapley NTU value)**.** For a strategic form game G, $u \in R^N$ is an α (β) Shapley NTU value if and only if there exists $x \in X$ such that $u_i = u_i(x)$ for all $i \in N$, there exists $\lambda \in \Delta^{n-1}$, $(\lambda_1 u_1, \cdots, \lambda_i u_i, \cdots, \lambda_n u_n)$ is a Shapley TU value of v_λ which is generated from α (β resp.) characteristic function V_α (V_β resp.) of G.

The next theorem shows that α/β Shapley NTU value is a natural extension of min-max (or max-min) value of two player zero-sum games. The similar result was already claimed by Luce and Raiffa(1957, p249) in the context of the Shapley value of TU games.

Theorem 1. *Let G be a two player zero-sum game, i.e., $N = \{1, 2\}$, $u_1(x) = -u_2(x)$ for all $x \in X$. If G has a Nash equilibrium, then there exists one and only one α (β) Shapley NTU value, which is equivalent to the min-max (and max-min) value of G.*

Proof. Define $v^\alpha_i \stackrel{\text{def}}{=} \max_{x^i \in X^i} \min_{x^j \in X^j} u_i(x^i, x^j)$ and $v^\beta_i \stackrel{\text{def}}{=} \min_{x^j \in X^j} \max_{x^i \in X^i} u_i(x^i, x^j)$ for each $i \neq j$. By assumption, $v^\alpha_i = -v^\beta_j = -v^\alpha_j = v^\beta_i$. Note that $V_\alpha(\{i\}) = \{u \in$

$R^2|u_i \leq v_i^\alpha\}$, and $V_\beta(\{i\}) = \{u \in R^2|u_i \leq v_i^\beta\}$ for $i \in N$. Thus,

$$\begin{aligned}\varphi_i(v_\lambda^\alpha) &= \frac{(1-1)!(2-1)!}{2\cdot 1}(v_\lambda^\alpha(\{i\}) - v_\lambda^\alpha(\emptyset)) + \frac{(1-1)!(2-1)!}{2\cdot 1}(v_\lambda^\alpha(\{j\}) - v_\lambda^\alpha(\{j\}))\\ &\quad + \frac{(2-1)!(2-2)!}{2\cdot 1}(v_\lambda^\alpha(\{i,j\}) - v_\lambda^\alpha(\{j\}))\\ &= \frac{1}{2}v_\lambda^\alpha(\{i\}) - \frac{1}{2}v_\lambda^\alpha(\{j\}) + \frac{1}{2}v_\lambda^\alpha(\{i,j\})\\ &= \frac{\lambda_i}{2}v_i^\alpha - \frac{\lambda_j}{2}v_j^\alpha + \frac{1}{2}v_\lambda^\alpha(\{i,j\})\\ &= \frac{1}{2}v_i^\alpha + \frac{1}{2}v_\lambda^\alpha(\{i,j\})\end{aligned}$$

Let $\lambda = (1/2, 1/2)$. Then $v_\lambda^\alpha(\{i,j\}) = 0$. Hence, $\varphi_i(v_\lambda^\alpha) = v_i^\alpha/2 = \lambda_i \cdot v_i^\alpha$. That is, (v_i^α, v_j^α) is an α (β) Shapley NTU value of G. Suppose that there exists $\lambda = (\lambda_i, 1-\lambda_i) \neq (1/2, 1/2)$, G has an α (β) Shapley NTU value $(u_i^*, -u_i^*)$. Then,

$$\varphi_i(v_\lambda^\alpha) = \frac{1}{2}v_i^\alpha + \frac{1}{2}(\lambda_i u_i^* + (1-\lambda_i)(-u_i^*)) = \lambda_i u_i^*,$$

where the last equality means an equation. Therefore $v_i^\alpha = u_i^*$. *Q.E.D.*

In a two-player zero-sum game, the outcome of Nash equilibrium is exactly min-max value. However, even if a two-player zero-sum game has no Nash equilibrium, α/β Shapley NTU value may exist. The next example represents this fact.

Example 1. Let G be a two player zero-sum game, whose utility function is the following matrix. Obviously G has no Nash equilibrium. Also, the α-Shapley NTU

$-1, 1$	$1, -1$
$1, -1$	$-1, 1$

value does not exist. However G has β-Shapley NTU values. Indeed, unfortunately, every outcome is a β-Shapley NTU value, which is supported by $\lambda = (1, 0)$ or $\lambda = (0, 1)$.

Lemma 2. *For a strategic form game G such that u_i is concave on X for each $i \in N$, let V be V_α (or V_β). Then α (or β) Shapley NTU value is α (or β) individually rational in the meaning of λ-transferred utility, that is, $\varphi_i(v_\lambda) \geq v_\lambda(\{i\})$ for every $i \in N$.*

Proof. By (3) of Lemma 1 the TU game v_λ is well defined, and is supper-additive, that is, $v_\lambda(S) + v_\lambda(T) \leq v_\lambda(S \cup T)$ for all $S, T \in \mathcal{N}$ where $S \cap T = \emptyset$ by (5) of Lemma 1. Thus, $v_\lambda(S \setminus \{i\}) + v_\lambda(\{i\}) \leq v_\lambda(S)$. Using this fact and the formula for the Shapley TU value, we can obtain the required result. *Q.E.D.*

Lastly, let us prove the existence of an α/β-Shapley NTU value.

Theorem 2. *For a strategic form game G such that u_i is concave on X for all $i \in N$, there exists an α/β-Shapley NTU value.*

Proof. Denote by D_λ the $(n \times n)$-diagonal matrix whose i-th diagonal element is λ_i. Define $\bar{v}_i \stackrel{\text{def}}{=} \inf_{\lambda \in \Delta^{n-1}} v_\lambda(\{i\})$. Set $M \stackrel{\text{def}}{=} \{u \in R^N | \forall i \in N, u_i \geq \bar{v}_i\}$, which is obviously convex, closed and bounded below. Set $V_\lambda(N) \stackrel{\text{def}}{=} (D_\lambda \cdot V(N)) \cap M$, which is non-empty, convex, compact for each $\lambda \in \Delta^{n-1}$.

In the light of Lemma 2, this trancation does not affect the existence of α/β-Shapley NTU value. The remains of the proof obviously follows from Shapley(1969, p262). See also Ichishi (1997,p333). *Q.E.D.*

In this theorem, the concavity of utility function does not seem so strong assumption. We believe that α/β Shapley NTU value may introduce a cooperative aspect, other than core, into well known models in strategic form games, e.g., oligopoly models or auction models.

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