Socially Subjective Equilibrium in Extensive Form Games *

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Abstract

This study describes a framework for the equilibrium analysis in the social science based on a new game theoretic equilibrium concept, *socially subjective equilibrium*(SSE), which was firstly introduced by Oishi (2007). In socially subjective equilibrium, each player views society via one's own solution concept. Each player's subjective image of society must be consistent with one's own observations for the realized outcome. In addition, each player's subjective image must satisfy a certain game theoretic solution concept, say, a social *norm*. These are the main features of our equilibrium concept. Socially subjective equilibrium is an equilibrium concept based on a behavioral hypothesis called *solution concept schemes* which we treat as an antipode to *best response schemes* . We extend the concept of SSE to extensive form games. In particular, we focus on Nash equilibrium, subgame perfect equilibrium, and sequential equilibrium as each player's concept used to arrive at a solution.

1 Introduction

In this paper, we extend the concept of *socially subjective equilibrium(SSE)* which was provided by Oishi (2007) into extensive form games. We will focus on the case that all players' solution concepts are Nash equilibrium, subgame perfect equilibrium, and sequential equilibrium. Especially, in the case of Nash equilibrium, we call the socially subjective equilibrium (SSE) *rationalized subjective equilibrium*, which was discussed by Oishi (2003) in the case of repeated games.

In the case that all players' solution concepts are Nash equilibrium, our framework is similar to other related studies; Kalai and Lehrer (1993)'s *Subjective Equilibrium*, Fudenberg and Levine (1993)'s *Self-Confirming Equilibrium*, or Hahn (1977)'s *Conjectural Equilibrium*. These papers have some common fashion. In all of above frameworks, players have a (maybe false) conjecture to the opponent players' strategies, which is consistent with equilibrium paths in a certain sense.

*The previous version of this paper forms a part of my doctoral thesis, which is titled "On socially subjective equilibirum".

However, there is a difference between the above equilibrium concepts and the SSE (or the rationalized subjective equilibrium). The above equilibrium concepts itself is a solution concept of a game. In the equilibrium, each player need not have a whole image of society and need not have a solution concept. In contrast, in an equilibrium of our framework, players must have some solution concept (e.g. Nash equilibrium, subgame perfect equilibirium, and so on), which need all players' rationality, and which may need subjective and consistent images of a game. While our framework relativizes many solution concepts, the settings of conjectural equilibrium, et al. provide only a solution concepts. This is the advantage of our framework and equilibrium concept.

In *socially subjective equilibrium* via Nash equilibrium, players have a coherence of their own norms in two senses. One is the coherency of the norm itself. Conjectured outcomes should satisfy some norm, that is, *Nash condition.* The other is the coherency between the imagined outcomes and realized outcomes. These are main features of our equilibrium concept.

The plan of this paper is as follows. Next section deals with subjective extensive games with (objectively) perfect information and defines *socially subjective equilibrium* in extensive form games. Section 3 will extend this concept to (objectively) imperfect information case. Both cases are an incomplete information game in subjective meanings. That is, each player does not know other players' payoff and can observe only plays on an equilibrium path. We also provide main theorems in Section 2 and 3. Theorem 2 and Theorem 3 claim the capability of our framework to describe a social state, where off-path-plays are usually different from what each member of society has imagined, and where each member subjectively accepts the social outcome by "stronger" solution concept than what can be objectively explained. Theorem 2 claims that in a perfect information game, every (objective) Nash equilibrium path can be supported by a certain subjective game and a (subjective) subgame perfect equilibrium. Theorem 3 says that in a game with imperfect information, every (objective) Nash equilibrium path can be supported by a certain subjective game and a (subjective) sequential equilibrium.

2 Perfect Information and Subjective Games

Let us demonstrate that the concept of socially subjective equilibrium concept can be extended to the extensive form game with (objectively) perfect information.

2.1 Notations and Definitions

In this paper, we will use some of the notations and definitions of Osborn and Rubinstein (1994). At first, we need define (*objective*) *extensive game* $\Gamma = (I,\ H,\ x,\ u_i(i \in I)\ ,S)$. However, we will omit the definition of *objective* extensive game since it is naturally defined from the definition of subjective game for player i. We shall let small letters in square brackets denote individual players who subjectively imagine each object below.

Definition 1 (Subjective Extensive Game for player i)**.** Let us define $\Gamma[i] = (I,\ H,\ x,\ u_j[i](i, j \in I)\ ,S)$ as a player i's *subjective* extensive form game, where $I = \{1, 2 \cdots, n\}$ is the set of players. H is the set of sequence (history) that satisfies that **[A]** $\emptyset \in H$ and **[B]** If $(a^t)_{t=1,\cdots,T} \in H$ and $L < T$, then $(a^t)_{t=1,\cdots,L} \in H$. We call the component of a history $h \in H$ an action taken by a player. A history $(a^t)_{t=1,\cdots,T} \in H$ is terminal if there is no a^{T+1} such that $(a^t)_{t=1,\cdots,T+1} \in H$. The set of terminal histories is denoted by Z.

The player function is defined as $x : H \setminus Z \to I$, where $x(h)$ is the player who takes an action after a history h. The player j's payoff function for player i is $u_j[i] : Z \to R$, which is assumed to be bounded. A pair (h, a) represents the history of length $k+1$ consisting of a k-length history followed by a.

Let us define the set of player i's node as $H_i \stackrel{\text{def}}{=} \{h \in H | x(h) = i\}$. And define the set of player i's action set at a history h as $A_i(h) \stackrel{\text{def}}{=} \{a | (h, a) \in H \text{ where } h \in H_i\}$. $S \stackrel{\text{def}}{=} \prod_{i \in I} S_i$ is the set of strategy profiles, where $S_i \stackrel{\text{def}}{=} \{s_i | s_i : H_i \to A_i(h)\}$ is the set of player i's strategy. A $s_j[i] \in S_j$ is the player j's strategy which player i expects for each $i, j \in I$, where $s_i[i] = s_i$ for $i \in I$. A profile $s[i] \in S$ denotes the strategy profile that player i expects. We assume that $u_i[i] = u_i$, that is, each player knows one's own payoff function.

We assume that every $h \in H$ has finite length. Hence every strategy profile $s \in S$ induces the unique terminal history $h \in Z$. A $p(s)$ denotes this induced history $h = (a^1, a^2, \cdots, a^T) \in Z$. We refer to the history $p(s)$ as the *induced path* below.

Assumption 1. *Every player cannot observe the opponent players' strategies. The players can only observe the sequence of actions, that is, the realized path.*

This assumption seems to be essential to our framework in this paper. In the case that all players' solution concepts are Nash equilibrium, if every player can also observe a profile of strategies, then our equilibrium concept is exactly (objective) Nash equilibrium .

Definition 2 (Subjective Extensive Game (SEG))**.** For an (objective) extensive game Γ define a subjective extensive game $\mathcal{G}$ as the profile of SEG for each player. That is $\mathcal{G} = (\Gamma[1], \Gamma[2], \cdots, \Gamma[n])$.

Next we define socially subjective equilibrium in subjective extensive games.

Definition 3 (Socially Subjective Equilibrium via Nash equilibrium (Extensive Form))**.** Let $\sigma^* = ((s_1^*, s_2^*[1], s_3^*[1], \cdots, s_n^*[1]), \cdots, (s_1^*[i], \cdots, s_i^*, \cdots, s_n^*[i]), \cdots, (s_1^*[n], \cdots, s_{n-1}[n], s_n^*))$ be a profile of subjective strategy profile. Define a strategy profile as $s^* \stackrel{\text{def}}{=} (s_1^*, s_2^*, \cdots, s_n^*)$. A profile σ^* is *a socially subjective equilibrium* via Nash equilibrium(SSE_{NE}) in $\mathcal{G}$ when it satisfies the following conditions;

(Nash Condition) $\forall i \in I$, $(\forall s_i \in S_i,\ u_i(p(s^*[i])) \geq u_i(p(s_i, s_{-i}^*[i]))) \wedge (\forall j \neq i, \forall s_j \in S_j,\ u_j[i](p(s^*[i])) \geq u_j[i](p(s_j, s_{-j}^*[i]))))$

(Consistency Condition) $\forall i \in I, p(s^*) = p(s^*[i])$.

The Nash condition means that each player has a belief consistent with Nash equilibrium in one's own subjective extensive game. That is, every player's solution concept is Nash equilibrium. However, even if each player have subjective rationality, it may be dogma. When the outcome confirms each player's rationality, each player's rationality and view of the game must have a coherency. That is, by the consistency condition, the observed equilibrium path must be consistent with the equilibrium path which each player has imaged subjectively.

There is not sufficient reason why all players should view an outcome of each game via the concept of Nash equilibrium. However, in the huge stock of past papers, we can find the reason why we should choose the Nash equilibrium as a player's solution concept.

2.2 Several Remarks

Remark 1. *Assume that σ^* is a SSE_{NE} of a subjective extensive game $\mathcal{G}$. The (subjective) equilibrium strategy profile $s^* \overset{\text{def}}{=} \prod_{i\in I} \mathrm{Proj}_{i\times i}\sigma^*$ [1] is not necessarily (objective) Nash equilibrium of the original (objective) extensive game Γ.*

The next example shows that this remark is true.

Example 1 (Chain-Store Game). We consider the chain-store game. The (objective) extensive game Γ is the following; $I = \{1, 2\}$; H consisting of the $\emptyset$,(IN),(OUT),(IN, CO) and (IN, AG); $x(\emptyset) = 1$ and $x((IN)) = 2$; $u_1((OUT))$=2,$u_1((IN, CO))$=3 and $u_1((IN, AG))$=0 ;$u_2((OUT))$=7,$u_2((IN, CO))$=3 and $u_2((IN, AG))$=0. A convenient representation of this game is shown as the tree in Figure 1. Note that this game has two Nash equilibrium

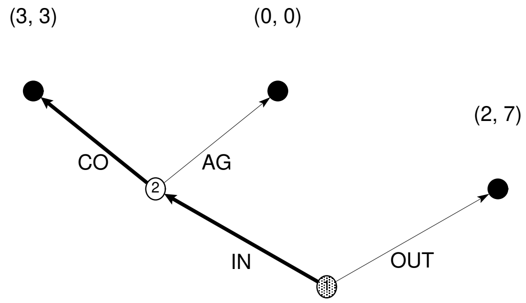

Figure 1: The original extensive form game : Γ

points (IN, CO) and (OUT, AG). We shall give SEGs for player 1 and 2. It is sufficient to define the imaged payoff functions in addition ; $u_2[1]((OUT)) = 7$,$u_2[1]((IN, CO)) = 2$,$u_2[1]((IN, AG)) = 4$, $u_1[2]((OUT)) = 4$,$u_1[2]((IN, CO)) = 3$ and $u_1[2]((IN, AG)) = 0$. Obviously the profile $\sigma^* = (s^*[1], s^*[2]) = ((OUT, \hat{AG}), (\hat{OUT}, CO))$ is SSE_{NE} of the $\mathcal{G}$. However, the (objective) strategy profile $s^* = (OUT, CO)$ is not (objective) Nash equilibrium of the Γ. Note that the realized path (OUT) is objective Nash equilibrium path of the objective game Γ with the strategy profile (OUT, AG) taken.

[1] $\mathrm{Proj}_k x$ means the projection of x to k-th coordinate.

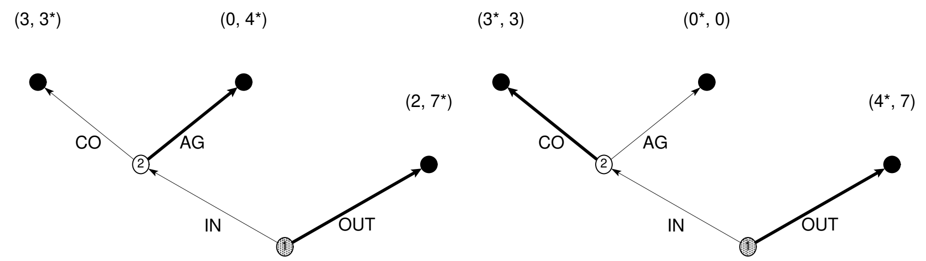

Figure 2: The subjective extensive form game for player 1 and 2 : $\mathcal{G} = (\Gamma[1], \Gamma[2])$

Fudenberg and Levine (1993) characterized Nash equilibrium by their weaker equilibrium concept, that is, self-confirming equilibrium. In the similar manner, we will demonstrate several remarks.

Theorem 1. *If a SEG $\mathcal{G}$ of Γ has a SSE_{NE} σ^*, then the realized path $h^* = p(\sigma^*)$ $= (a^{1^*}, a^{2^*}, \cdots, a^{T^*})$ is an (objective) Nash equilibrium path of the original (objective) extensive game Γ.*

Proof. Assume that the path h^* is not the Nash equilibrium path of the Γ. Define a set S^* as the set of strategies which induces the path h^*, that is, $p(s^*) = h^*$ for all $s^* \in S(h^*)$. Then $\forall s^* \in S(h^*)$, $\exists i \in I$, $\exists s_i \in S_i, u_i(p(s_i, s^*_{-i})) > u_i(p(s^*))$. Note that this deviation must induce the path which is different from the path h^*.

This observed deviation from the realized path h^* is also observable in any SEG $\mathcal{G}$ of Γ which induces the path h^*. Therefore, each strategy $s^* \in S(h^*)$ also fails to satisfy the Nash condition of SSE_{NE}. This is a contradiction. *Q.E.D.*

We can also make sure that even if a realized path is what is induced by a (objective) subgame perfect Nash equilibrium of Γ, the objective strategy profile of the SSE_{NE} in some $\mathcal{G}$ is not necessarily (objective) Nash equilibrium of Γ. Conversely next remark also holds.

Remark 2. *Suppose that $\sigma^* = (s^*[1], \cdots, s^*[n])$ is a SSE_{NE} of a $\mathcal{G} = (\Gamma[1], \cdots, \Gamma[n])$. In addition, suppose that for all $i \in I$, $s^*[i]$ is a (subjective) subgame perfect Nash equilibrium of the $\Gamma[i]$. That is, each player i's solution concept assumes to be subgame perfect Nash equilibrium. However, the realized path $p(\sigma^*)$ is not necessarily the (objective) subgame perfect Nash equilibrium path of the (objective) game Γ.*

The following example is a proof of Remark 2.

Example 2. The original extensive game is the Γ where $I = \{1, 2\}$; H consisting of the $\emptyset$,(L),(R),(L, A),(L, B),(L, A, a),(L, A, b),(L, B, c) and (L, B, d); $x((L)) = 2$ and $x(h) = 1$ for every non-terminal history $h \neq \emptyset$; $u_1((R)) = 0$,$u_1((L, A, a)) = 2$,$u_1((L, A, b)) = 3$,$u_1((L, B, c)) = 4$ and $u_1((L, B, d)) = 1$; $u_2((R)) = 2$,$u_2((L, A, a)) = 4$,$u_2((L, A, b)) = 7$,$u_2((L, B, c)) = 5$ and $u_2((L, B, d)) = 2$. The game Γ has three Nash equilibrium points (Lbc, A), (Lbd, A) and (Lac, B). Furthermore one can easily confirm that the profile

(Lbc, A) is the unique subgame perfect Nash equilibrium of Γ. When the profile (Lbc, A) attains, the realized path is the sequence (L, A, b).

Next we will provide SEGs for player 1 and 2. Give the imaged payoff functions as $u_2[1]((R))$= 2,$u_2[1]((L, A, a))$= 4,$u_2[1]((L, A, b))$= 3, $u_2[1]((L, B, c))$= 5,$u_2[1]((L, B, d))$= 2, $u_1[2]((R))$= 0,$u_1[2]((L, A, a))$= 3,$u_1[2]((L, A, b))$= 2, $u_1[2]((L, B, c))$= 4 and $u_1[2]((L, B, d))$= 1. A convenient representation of these subjective extensive games $\Gamma[1]$ and $\Gamma[2]$ are also shown as the trees in Figure 3; the left is the player 1's and the right the other's.

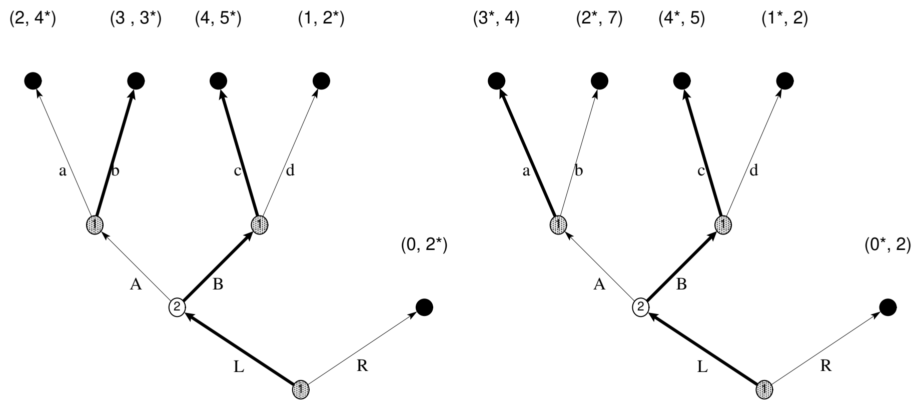

Figure 3: Subjective Extensive Form Games for Player 1 and 2: $\Gamma[1]$ and $\Gamma[2]$

We can observe that the $\Gamma[1]$ has the unique subgame perfect Nash equilibrium $s^*[1] = (Lbc, \hat{B})$ and that the $\Gamma[2]$ has the equilibrium $s^*[2] = (L\hat{a}c, B)$ also. Clearly $\sigma^* = ((Lbc, \hat{B}), (L\hat{a}c, B))$ is a RSE of the $\mathcal{G}$. However the path $p(\sigma^*) = (L, B, c)$ is not the subgame perfect Nash equilibrium path of the Γ.

We can restate and generalize the above example as the following theorem.

Theorem 2. *For every Γ, for every (objective) Nash equilibrium path h^*, there exists a subjective extensive form game $\mathcal{G}$, the path h^* can attains as a (subjective) subgame perfect Nash equilibrium of $\Gamma[i]$ for all $i \in I$.*

Proof. Assume that a strategy profile $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ which induces a Nash equilibrium path h^*. For $\Gamma[i]$, we can construct $j \neq i$'s player's payoff function imaged by player i as the following;

$$u_j[i](s[i]) = \begin{cases} 1 & \text{for any } s[i] \text{ where } p(s[i]) = h^* \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is obvious that any strategy $s_j[i]$ which induces the equilibrium path h^* is best response to each subgame for all $j \in I \setminus \{i\}$. Fix $s_j[i]$ as s_j^* for all $j \neq i$. Note that on the equilibrium path h^* each action is best response to each subgame. Hence it is satisfactory to find actions a_i^* which is best response to each subgame off the equilibrium path and to check that the action profile, that is, the strategy is also best response to each subgame on the equilibrium path.

By the backward induction, we can find player i's actions which is best response to $\{s_j[i]\}_{j\in I\setminus\{i\}}$ for each subgame off the equilibrium path. If the actions on the equilibrium path is not best response to this constructed (subjective) actions off the equilibrium, it is contradiction to the fact that s_i is (objective) Nash equilibrium strategy. Therefore these constructed (subjective) strategies are also (subjective) subgame perfect equilibrium of player i's subjective game $\Gamma[i]$. We can apply these construction to each $\Gamma[i]$ and $s[i]$ for $i \in I$. Then $\sigma^* = (s[1], \cdots, s[n])$ induces the original path h^* and $s[i]$ is (subjective) subgame perfect Nash equilibrium of $\Gamma[i]$ for all $i \in I$. *Q.E.D.*

This theorem means that every (objective) Nash equilibrium path can be supported by (subjective) subgame perfect equilibrium. That is, each player can recognize an *objective* Nash equilibrium outcome as a *subjective* subgame perfect equilibrium.

3 Imperfect Information and Subjective Games

As we have discussed, our framework is restricted to (objectively) perfect information games. However we can extend our equilibrium concept to (objectively) imperfect information games if we adopt naive attitude toward the meaning of "consistency". Our "consistency" means that all players' conjectured equilibrium paths are coherent with an *ex-post* realized path. Therefore we can also define the consistency as *the coherency between the equilibrium probability distribution over all paths, which each player imagines, and the realized distribution.* While we consider that this consistency requires support by some learning process, we don't discuss it here because it is not our main object.

3.1 Definitions

We will extend the concept of subjective game to (objectively) imperfect information case without chance move.

Definition 4 (Subjective Extensive Game for player i)**.** Let us define $\Gamma[i] = (I, H, S, \{\mathcal{P}_i\}_{i\in I}, \{U_j[i]\}_{j\in I})$ as a subjective extensive game, where $I = \{1, 2, \cdots, n\}$ is the set of players. H and $A_i(h)$ are defined in the same way of Definition 1. The player function is defined as $x : H \setminus Z \to I$. For each player $i \in I$, there exists a partition $\mathcal{P}_i$ of $H_i \overset{\text{def}}{=} \{h \in H | x(h) = i\}$ with the property that $\forall P_i \in \mathcal{P}_i$, $\forall h, h' \in P_i$, $A_i(h) = A_i(h')$. For $P_i \in \mathcal{P}_i$, $A_i(P_i)$ denotes the set $A_i(h)$ for $h \in P_i$. $S \overset{\text{def}}{=} \prod_{i\in I} S_i$ is the set of (behavioral) strategy profiles, where $S_i \overset{\text{def}}{=} \{s_i | s_i : \mathcal{P}_i \to \Delta(A_i(P_i))\}$ is the set of the player i's strategy, where $\Delta(A_i(P_i))$ is the set of probability measures $\mu_i(A_i(P_i))$ on $A_i(P_i)$ for each $P_i \in \mathcal{P}_i$. A $s_j[i] \in S_j$ is the player j's strategy which player i expects for each $i, j \in I$, where $s_i[i] = s_i$ for $i \in I$. Define $\mu(h) \overset{\text{def}}{=} \prod_{i\in I, A_i(P_i)\cap h\neq\emptyset, a\in A_i(P_i)\cap h} \mu_i(A_i(P_i))(a)$, where $A(P_i) \cap h \overset{\text{def}}{=} \{a \in A(P_i) | a = \text{Proj}_k h \text{ for some } k\}$. For every s_i, we can naturally induce probability measures $\mu_i(\cdot)$. Therefore a payoff function can be defined as $U_i(s) \overset{\text{def}}{=} \sum_{z\in Z} u_i(z)\mu(z)$. $\mu_+(s)$ denotes the sequence of probability measures, where $\mu_+(s) \overset{\text{def}}{=}$

$\{\mu(h)\}_{h\in\{h'\in H|\forall k,\exists k',\mathrm{Proj}_k h'=\mathrm{Proj}_{k'} z,\mu(z)>0\}}$. We refer to the measures $\mu_+(s)$ as the *induced distribution* below.

An (objective) extensive game $\Gamma = (I, H, S, \{\mathcal{P}_i\}_{i\in I}, \{U_j\}_{j\in I})$ is naturally defined in the obvious way. The next definition of SSE_{NE} is a natural extension of Definition 3.

Definition 5 (Socially Subjective Equilibrium via Nash equilibrium (Extensive Form))**.** Let $\sigma^* = ((s_1^*, s_2^*[1], s_3^*[1], \cdots, s_n^*[1]), \cdots, (s_1^*[i], \cdots, s_i^*, \cdots, s_n^*[i]), \cdots, (s_1^*[n], \cdots, s_{n-1}[n], s_n^*))$ be a profile of subjective strategy profile. Define a strategy profile as $s^* \stackrel{\text{def}}{=} (s_1^*, s_2^*, \cdots, s_n^*)$. A profile σ^* is *a socially subjective equilibrium* in $\mathcal{G}$ when it satisfies the following conditions;

(Nash Condition) $\forall i \in I, (\forall s_i \in S_i,\ U_i(s^*[i]) \geq U_i(s_i, s_{-i}^*[i])) \wedge (\forall j \neq i,\ \forall s_j \in S_j,\ U_j[i](s^*[i]) \geq U_j[i](s_j, s_{-j}^*[i]))$

(Consistency Condition) $\forall i \in I, \mu_+(s^*) = \mu_+(s^*[i])$.

3.2 A Result

We must notice that the concept of socially subjective equilibrium via Nash equilibrium in the imperfect information case is meaningless when all paths are reachable with a positive probability. That is, if for all $i \in I$, s_i is completely mixed and if σ is a SSE_{NE} of $\mathcal{G}$, then the corresponding strategy profile s^* is the Nash equilibrium of the original (objective) game Γ. However, in imperfect information games the example of "Selten's horse" shows that Remark 1 holds in the case that some strategy is not completely mixed.

Example 3 (Selten's Horse (Selten 1975)). Let an (objective) game Γ be the well-known Selten's horse. Subjective games for player 1,2 and 3 are defined in Figure 4. The left is the player 1's subjective game and the right the others'. SSE_{NE} with the realized path (D, L) is also (subjective) sequential equilibrium of each $\Gamma[1]$, $\Gamma[2]$ and $\Gamma[3]$. However, the (objective) strategy profile (D, d, L) is not (objective) Nash equilibrium of Γ.

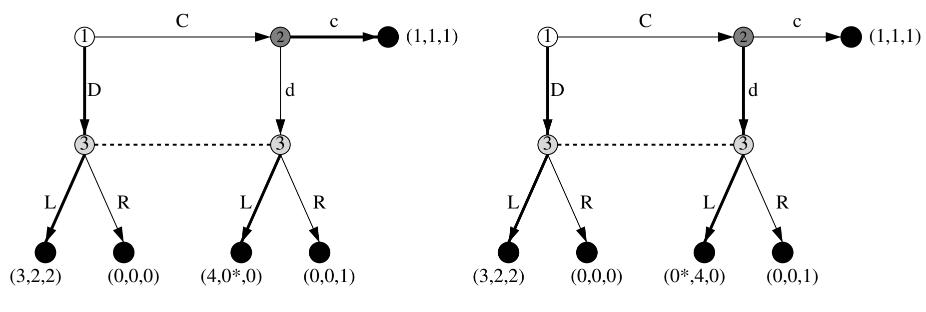

Figure 4: Subjective Games for Player 1,2 and 3 : $\Gamma[1]$ and $\Gamma[2] = \Gamma[3]$

In contrast to the case of perfect information, we can find that the path of SSE_{NE} is not necessarily Nash outcome in the case of imperfect information. This is shown in the example of Fudenberg and Levine (1998, p177).

Next theorem shows that every Nash equilibrium distribution can be supported by (*subjective*) *sequential equilibrium* for all players. In other words, every Nash equilibrium path is induced by some subjective game and SSE_{NE} which is refined by replacing Nash (equilibrium) condition with sequential equilibrium condition.

Theorem 3. *For every* Γ*, for every Nash equilibrium distribution* μ^*_+*, there exists a subjective extensive form game* $\mathcal{G}$*, the induced distribution* μ^*_+ *can attains as a sequential equilibrium of* $\Gamma[i]$ *for all* $i \in I$*.*

Proof. Note that we omit players' indexes of several notions for convenience. Fix a strategy profile $s = (s_1, s_2, \cdots, s_n)$ which induces Nash equilibrium distribution μ^*_+. For $\Gamma[i]$ one can construct $j(\neq i)$ player's payoff function, strategy $s_j[i]$, and the *beliefs* through the following procedure.

STEP 0 Define $j(\neq i)$ player's (subjective) payoff functions conjectured by player i as

$$u_j[i](z) = \begin{cases} 1 & \text{for any } z \text{ where } \mu(z) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

STEP 1 For all $j' \in I$, for all $P_{j'} \in \mathcal{P}_{j'}$ where $\mu(h) > 0$ for some $h \in P_{j'}$, give a probability measure (*belief*) b on each $P_{j'}$ by the rule of Bayes, i.e., $b(h) \overset{\text{def}}{=} \mu(h)/(\sum_{h' \in P_{j'}} \mu(h'))$. Especially $P^b_{j'}$ denotes each $P_{j'}$ given some belief b. Let us denote the initial node by P^*.

STEP 2 Find a $P_{j'} \in \mathcal{P}_{j'}$ such that $P_{j'} \neq P^b_{j'}$ where for all $h \in P_{j'}$ $\mu(h) = 0$ and for some $h' \in H$ where $\mu(h') > 0$, for some $a \in A(h')$, $h = (h', a)$. This factor of partition is denoted by P^0. If there is not such P^0, skip this step and go to **STEP 4**.

STEP 3-0 When $\forall k$, $\exists k'$, $\text{Proj}_k h = \text{Proj}_{k'} h'$, we can naturally define a "conditional" probability measure as $\mu(h'|h) \overset{\text{def}}{=} \prod_{i \in I, A_i(P_i) \cap (h' \setminus h) \neq \emptyset, a \in A_i(P_i) \cap (h' \setminus h)} \mu_i(A_i(P_i))(a)$, where $h' \setminus h = \prod_{\forall k, \text{Proj}_{k'} h' \neq \text{Proj}_k h} \text{Proj}_{k'} h'$.

If $\forall h \in P^0$, $\forall a \in h$, $\forall j' \in I$, $a \notin A(P^b_{j'})$, then give any probability measure (*belief*) b on P^0, else define a belief b as the following [2]. In this case, there exists $a \in h \in P^0$, $a \in A(P^b_{j'})$. For all such a, fix the probability $\mu(A(P^b_{j'}))(a)$. For all other a such that $a \in h \in P^0$, for sufficiently small $\epsilon > 0$ define a probability measure as

$$\mu^\epsilon(A(P))(\cdot) = \begin{cases} \mu(A(P))(\cdot) - \epsilon(|A(P)|/|A^+(P)| - 1) & \text{if } \mu(A(P))(a) > 0 \\ \epsilon & \text{otherwise} \end{cases}$$

[2] $a \in h$ means that there exists k, $a = \text{Proj}_k h$.

where $A^+(P) \stackrel{\text{def}}{=} \{a \in A(P) | \mu(A(P))(a) > 0\}$, i.e., the support of $\mu(A(P))$. Under these probability measures, give a belief b on P^0 via Bayes' rule. That is,

$$b(h) \stackrel{\text{def}}{=} \frac{\sum_{h' \in P^*} \mu^\epsilon(h|h')b(h')}{\sum_{h'' \in P^0} \sum_{h' \in P^*} \mu^\epsilon(h''|h')b(h')}$$

STEP 3-1 Define a restricted strategy set $S_i(P^0) \stackrel{\text{def}}{=} \{s_i(P^0) | s_i(P^0) : \mathcal{P}_i|_{P^0} \to \Delta(A_i(P_i))\}$, where $\mathcal{P}_i|_{P^0} \stackrel{\text{def}}{=} \{P_i \in \mathcal{P}_i | P_i \neq P_i^b$ or $P_i = P^0$, i.e., P_i is not given any belief yet or P^0 itself $\}$. $Z|_{P^0}$ denotes the set of all reachable terminal nodes from P^0. Define an expected payoff for this strategy set as $E_i(s_i(P^0)) \stackrel{\text{def}}{=} \sum_{h \in P^0} \sum_{z \in Z|_{P^0}} u_i(z)\mu(z|h)b(h)$.

STEP 3-2 Choose $s_i(P^0) \in S_i(P^0)$ which maximizes $E_i(s_i(P^0))$. If $S_i(P^0) = \emptyset$, skip this step.

STEP 3-3 For all $j' \in I$, for all $P_{j'} \in \mathcal{P}_{j'}$ where $\mu(h|h^0) > 0$ for some $h \in P_{j'}$ and for some $h^0 \in \{h \in P^0 | b(h) > 0\}$, give a belief b on each $P_{j'} (\neq P_{j'}^b)$ by the Bayes' rule, i.e.

$$b(h) \stackrel{\text{def}}{=} \frac{\sum_{h' \in P^0} \mu(h|h')b(h')}{\sum_{h'' \in P_{j'}} \sum_{h' \in P^0} \mu(h''|h')b(h')}.$$

STEP 3-4 Find a $P_{j'} \in \mathcal{P}_{j'}$ such that $P_{j'} \neq P_{j'}^b$ where $\forall h \in P_{j'}$, $\forall h' \in P^0$ $\mu(h|h') = 0$ and $\exists h'' \in H$, $\exists h' \in P^0$, $\exists a \in A(h'')$, $\mu(h''|h') > 0$ and $h = (h'', a)$. If one can find such $P_{j'}$, re-define it as the new P^0, re-define the original P^0 as the new P^* and go back to **STEP 3-0**, else define P^* as the initial node again and go to **STEP 2**.

STEP 4 End.

For other players, construct subjective games, strategies, and beliefs in the same way. One can check that the beliefs satisfy Kreps and Wilson (1982)'s *consistency* by **STEP 1**, **3-0** and **3-3** (continuity of beliefs) and that the strategies are *sequentially rational* by **STEP 0** and **3-2**. Therefore the constructed strategy profile $s[i]$ is sequential equilibrium of this constructed $\Gamma[i]$ for all player i. Also, the (subjective) strategies induces the same distribution as the objective strategy profile s does. *Q.E.D.*

In this theorem, we cannot replaces *sequential equilibrium* with *trembling hand perfect equilibrium.* Next example shows that a (objective) sequential equilibrium path can never be subjectively supported by (subjective) trembling hand perfect equilibrium of any $\Gamma[i]$.

Example 4. Figure 5 is the example which shows sequential equilibrium is not necessarily (trembling hand) perfect equilibrium in Kreps and Wilson (1982, p883). One can easily check that the realized path (L, r) fails to be viewed as (subjective) perfect equilibrium of any subjective game $\Gamma[1]$.

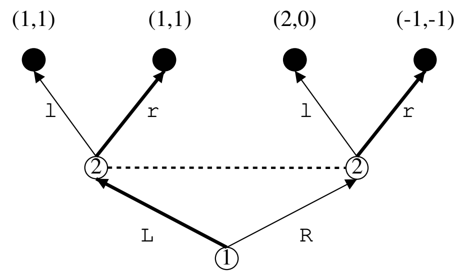

Figure 5: The example of Kreps and Wilson (1982)

4 Discussion

The implications of Theorem 2 and 3 are important. Even if every player has *subgame perfect Nash equilibrium* or *sequential equilibrium* as solution concept of society or norm of one's own satisfaction, any Nash equilibrium path can be subjectively supported by such norms. This is because our equilibrium concept allows only consistency between realized paths and imagined paths of each player. Conjectured off-path-plays of players may not have the consistency. Consequently, "stronger" solution concepts depending on off-path plays will not refine Nash equilibrium. In other words the "stronger" solution concept is confirmed by the outcome induced through the "weaker" solution concept. Even if we can describe players' behaviors via only weak and objective solution concept, we can explain their behavior via stronger and subjective solution concept.

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