Decentralized Exchange Economy with the strategic allocations of initial endowments

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Abstract

In this paper, we provide a decentralized pure exchange model, which includes each player's strategic decision-making about to which market the player should divide one's own initial endowment and how much the player also allocates it. We will define two concepts of equilibrium, that is, the Walras equilibrium and the Nash-Walras equilibrium in the decentralized pure exchange economy. We will also prove the existence of the equilibrium and the coincidence between the Walras equilibrium in the centralized economy and the Nash-Walras equilibrium in some decentralized market. For three players who possess equal initial endowments, we will provide numerical examples for six characteristic market structures.

1 Introduction

In this paper, we provide a decentralized pure exchange model, which includes each player's strategic decision-making about to which market the player should divide one's own initial endowment and how much the player also allocates it. These strategic selections of initial endowments are the main feature and contribution of our coalitional model. In the numerical examples in Section 4, the interesting results of our model will be shown. In the examples, we can recognize that our models are simply computationally designed for complex situations.

Our concept of a "decentralized" model includes a "centralized" model of a single coalition, a network model consisting of only two-player coalitions, and a coalitional market model consisting of three or more player coalitions. The decision process of our model is one shot game. That is, the equilibrium price and the initial endowment allocations may be decided simultaneously.

Some players may be strategic, other players may not be strategic about planning initial endowment allocations in each market structure. On the other hand, each player is always a price taker because we assume that the market mechanism is not directly controlled by small players. However, each player can recognize the market mechanism itself in the long term. Each player can also increase its utility by choosing the initial endowment allocation plan. That is, each player can indirectly affect the equilibrium price. These are our settings.

Therefore, our aim is not the realization of the Walras equilibrium by the Nash equilibrium, for example, Schmeidler [24], Ghosal and Polemarchakis [11], and so on, although we will introduce both equilibrium concepts. As Ostroy and Starr [19] , one of the most famous articles on the decentralized market model, told the story of the exchange market in their article, we also provide a similar situation of the decentralized market.

As Shapley and Shubik [1] explain, in daily life, we mutually trade some goods with some individuals in some markets. For example, a trader has two certain amounts of money. One is the dollar. The other is the euro. In general, the currency market is decentralized without a central authority, including interactions between individuals, groups, or coalitions. Each submarket will decide the equilibrium exchange rate of the currency. In the decentralized market, there may possibly be different exchange rates for currencies or goods; that is, different prices.

These diversities of combinations are revealed not only among individuals, but also among groups. From the perspective of sociology, for example, Giddens and Sutton [12, Chapter 11:Stratification and Class], our circumstances are said to be constrained by not only economic structure, but also cultural, religious, occupational, and other social structures. All consumers do not exchange at one specific stratification. The members of these groups are not necessarily exclusive. In general, the boundaries between groups, classes, and stratification are vague.

We will formalize these complex situations as each player belongs to multiple coalitions simultaneously, which are not necessarily "partitioned". In other words, our model involves the situation of coalitional structure in the meaning of cooperative game theory.

We assume that each player makes a decision in two phases. In the stage of initial endowment plannings, the player will choose the amounts of initial endowment for each market that the player is concerned about. The player will behave strategically under other players' initial endowment, forecasting the following adjustment of exchange to market equilibrium. In the stage of market coordination, the player maximizes one's utility given the prices of each market. After the process of adjustment between demand and supply, a market equilibrium will be achieved. The player must make a strategic decision until the other players reach strategic maximization. In the final state, the economy may be the Nash equilibrium in the sense of strategy for initial endowment allocations and the Walras equilibrium in the sense of market clearings.

In the literature on the decentralized economy, our model may be a slight variation regarding the amounts of genuine articles, for example, Ostroy and Starr [19], Shapley and Shubik [25], Rubinstein and Wolinsky [20], and so on. Our model may also extend a network model as a special case, which occurs when all markets are doubleton, that is, $\{1,2\},\{2,3\},\{1,3\}$,$\{3,4\}$, ... , etc. Of course, we recognize the vast amount of articles about network models, and we can find a list of profound literature, for example, Jackson [13] and [14], Sargent and Stachurski [23], or, as one of the recent superior research efforts, Cassese and Pin [6], which address fairness in the pure exchange economy within the network model. The goal of this article may not be to add a page to the vast volume of these sophisticated papers, namely networking exchange models or network formation models.

In the literature on equilibrium manipulation, our model may also be an extension of these manipulation models. Manipulation models would suggest that changes in the allocation of initial endowments can change the equilibrium points. In the model of Safra [21], which assumes differentiable utility functions, and assumes that the market structure may consist of the whole coalition and all singleton coalitions, for example, $\{1,2,3\},\{1\},\{2\},\{3\}$. The other equilibrium manipulation model, Safra [22], also provides similar coalitional situations, where each player mutually affects common submarkets. This is because the aim of their models is to reveal the effectiveness and relationship between a market equilibrium and strategic allocations of initial endowment.

However, our purpose is another direction of research. We will suggest the importance of the fact that the not only strategic allocation of initial endowment, but also diversities of the coalitional combinations affect the equilibrium points.

From the aspect of the oligopolistic behavior of players in the Walras equilibrium, many fruitful models and discussions have been made. For example, d'Aspremont et al. [7], Gabszewicz [10], Dickson and Tonin [8], etc. In some of these papers, for example, a part of the players may be oligopolistic, and the others may not affect the equilibrium price directly. The economies of these models are the mixture of several explicit roles.

However, all players in our model may behave under the common role or the common decision rule. The difference in the roles of the players may be implicitly generated by the market structure. That is, with whom does the player communicate? Who trades with players? Who is a member of the coalitions? A player may be a member of a single coalition. Another player may exchange commodities in many trading groups. The linking or grouping style of the players is variable.

The plan of this article is as follows. The next section will introduce the definition of the market structure and the decentralized pure exchange economy. We will also prove the relevant claims. In Section 3, we will define two concepts of equilibrium, that is, the Walras equilibrium and the Nash-Walras equilibrium in this decentralized pure exchange economy, as stated above. We will prove the existence of both equilibria in any market structure. Theorem 5 shows the relationship between the two equilibrium concepts.

The section 4 shows six numerical and interesting examples. The first is the case where each player is not connected with anyone. The second is the case where two players are linked with each other, and one of the two players can consume a part of the initial endowment without providing it to the belonging market, that is, self-consumption. In the third case, the other player linked with the player can also make self-consumption in addition to the second case. The fourth case provides the usual equilibrium of the centralized economy. The fifth case is the situation where one player is the "middleman." In the sixth case, the economy is fully networked. All players may link up with each other. That is, this network is a "complete graph." In the last section, we will provide concluding remarks.

2 Definitions

2.1 The Market Structure

Let us define a market structure below. For our definition of the market structure, we have referred to Malamud and Rostek [15], which sophisticatedly analyze decentralized markets of risky assets, in contrast to our model, which treats multiple and non-risky goods.

$N = \{1, 2, \cdots, n\}$ is the finite set of players(or consumers). A set $\mathcal{P}(N)$ represents a set of all subsets of N. That is, we define $\mathcal{P}(N) = \{M | M \subset N\}$ as a market set, which is a power set of N. A set of market structures $\mathcal{MS}$ is defined by $\mathcal{MS} = \{\mathcal{M} | \cup_{M \in \mathcal{M}(N) \subset \mathcal{P}(N)} M = N\}$. Each factor $\mathcal{M} \in \mathcal{MS}$ is also called a *market structure*. Therefore, each player in N must belong to some markets in a market structure. Each $M_j \in \mathcal{M}, j = 1, 2, 3, \cdots, m$ is a market to which several players belong or to which only one player belongs. An m is a finite number of markets. Therefore, a market structure shows a bundle of trade places belonging to each player. Throughout this paper, the market structure is given. We do not discuss the formation of market structures. We would focus on the steady state of the market in the situation in which the player accepts the strategic choice of endowment for each market.

Consider the case of $N = \{1, 2, 3\}$. For example, we can consider some market structures $\mathcal{M} = \{M_1, M_2, M_3, M_4\}, M_1 = \{1, 2, 3\}, M_2 = \{1, 2\}, M_3 = \{1, 3\}$ and $M_4 = \{3\}$. The following Figure 1 represents this market structure $\mathcal{M}$.

The market M_1 is a centralized market to which all players belong. However, player 1 also belongs to the market M_2 and M_3 at the same time. These are decentralized markets where player 1 can trade or exchange with player 2 or 3. The market M_4 is a singleton set, where there is no trade. Player 3 should decide simultaneously how much goods one will consume by oneself in the "market" M_4 and how much goods one will provide for the centralized market M_1 and the decentralized market M_3. In the market M_3, player 3 should directly negotiate with player 1 through a process of coordination.

In this example, a centralized market usually consists of only one ground coalition N, that is, the market M_1. Therefore, our decentralized modeling includes a centralized market exchange model. If the market structure only consists of a pair of two players, our model coincides with the network models.

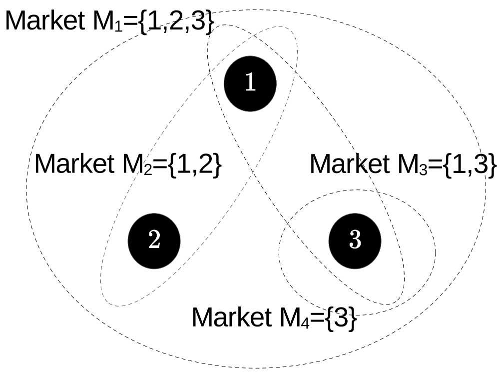


Figure 1: A market structure

2.2 The Decentralized Pure Exchange Economy

In this paper, we consider a pure exchange economy in a market structure $\mathcal{M}$.

R^l represents a l-dimensional Euclidean metric space. [1] A set $X_i = \{\sum X_i^M | X_i^M \subset R^l, i \in M \in \mathcal{M}\}\}$ is defined as the set of consumption of the player i for each $i \in N$.

Let $\omega_i \in R^l_+ \backslash \{0\}$ be a player i's initial endowment. We assume that $\sum_{i \in N} \omega_i = \omega \in R^l_{++}$ We will also define the set of initial endowment strategy of the player i' for a market structure $\mathcal{M}$ as follows:

$$\Omega_i = \left\{ \prod_{i \in M \in \mathcal{M}} \omega_i^M \middle| \sum_{i \in M \in \mathcal{M}} \omega_i^M = \omega_i, \omega_i^M \in R_+ \right\}$$

We can also define the strategy set of initial endowments as $\Omega = \prod_{i=1}^n \Omega_i$.

Lemma 1. *The set of strategy for initial endowments Ω is a non-empty, compact, and convex set.*

Proof. We can easily prove the nonemptiness of Ω. At least, we can always construct a strategy as follows. For some $\omega_i^{\mathcal{M}} \in \Omega_i$,

$$\omega_i^{\mathcal{M}} = \Big(\frac{\omega_i}{|\{k | i \in M_k \in \mathcal{M}\}|}, \cdots, \frac{\omega_i}{|\{k | i \in M_k \in \mathcal{M}\}|} \Big).$$

$|A|$ means the cardinality of the set A. In Euclidean space, the compactness of the set is equivalent to the condition that the set is bounded and closed. First, we will prove the boundedness of Ω. Obviously, for Ω_i of each i, the $(0, \cdots, 0)$(K-tuples,$K = |\{k|i \in M_k \in \mathcal{M}\}|$) is an infimum and also $(\omega_i, \cdots, \omega_i)$($K$-tuples) is a supremum. Therefore, Ω_i is bounded above and below.

Second, we will show that Ω_i is closed. Suppose that a sequence $\{\omega_i^n\}(\omega_i^n \in \Omega_i, n = 1, 2, \cdots)$ is a convergent sequence such that $\lim_{n \to \infty} \omega_i^n = \omega_i^*$. For each n, there exists some $\bar{j}$, and $\omega_i^n = (\omega_i^{M_1 n}, \cdots, \omega_i^{M_j n}, \cdots, \omega_i^{M_{\bar{j}} n})$ satisfies $\sum_{j \in \{k | i \in M_k \in \mathcal{M}\}} \omega_i^{M_j n} = \omega_i$ by definition. Hence, the limit point ω_i^* also satisfies $\sum_{j \in \{k | i \in M_k \in \mathcal{M}\}} \omega_i^{M_j *} = \omega_i$. Therefore, Ω_i is closed and then compact.

[1] In general, every Euclidean metric space is a Hausdorff space. And every compact Hausdorff space is a Baire space (Munkres[18, Theorem 48.2(Baire categoy theorem)]. The main article(Dubey and Ruscitti[9]) to which we will refer proved the main theorem in the locally Hausdorff space and the Baire space. However, our goal is not the generality of topological space. Therefore, in this paper, all mathematical space is a Euclidean metric space without several finite sets.

Next, we will prove the convexity of Ω_i. Suppose that $\omega_i^a, \omega_i^b \in \Omega_i$. However, we can also easily prove that $\alpha \cdot \omega_i^a + (1-\alpha) \cdot \omega_i^b$ also satisfies the above equation for any $\alpha (0 \leq \alpha \leq 1)$ because we know that $\alpha \cdot \omega_i + (1-\alpha) \cdot \omega_i = \omega_i$ for any ω_i^a and ω_i^b, for any α. Lastly, we can prove the lemma by applying the fact that Ω is finite products of Ω_i for all $i \in N$. *Q.E.D.*

Let $\Delta^j = \{p^j \in R_+^l | p^j = (p^{1j}, p^{2j}, \cdots, p^{lj}), \sum_{k=1}^{l} p^{kj} = 1\}$ be the price set for a market M_j. The element $p^j \in \Delta^j$ is the price of a market M_j. Let $\Delta^{\mathcal{M}} = \prod_{j=1}^{m} \Delta^j$ be the set of prices for a market structure $\mathcal{M}$. We should remark that these settings, that is, consumption sets and prices, are defined over the market structure, not the each market. Our pricing rule requires that multiple price systems be synthesized through each player to maximize their own utility, which may amount their own offers in the relative markets.

Assumption 1. If $\exists M \in \mathcal{M}, |M| = 1$,then $p^M = (p^{M1}, \cdots, p^{Ml}) = p_{++}^* > 0$.

This dummy price p_{++}^* is any positive price, the notation of which is commonly used in this article. Because the market of our model includes the single player case, we need Assumption 1, which provides some dummy prices. If some market M is a singleton coalition, the single player on the market M cannot trade with anyone. The equilibrium price of the market is not endogenously determined. However, if the price of some good is zero, the player can consume infinite amounts of the good. This is not an equilibrium. Therefore, Assumption 1 is required for the market equilibrium of a singleton market. This assumption may mean that each player knows the finiteness of the goods as some imagined positive price.

Next, the utility of each player *i*' is defined as a function $u_i : X_i \to R$.

Assumption 2. The utility function of each player *i*' u_i is continuous, strictly quasi-concave, and strictly monotonic.

A pure exchange economy in a market structure is defined by $\mathcal{E}(\mathcal{M}) = \{N, \mathcal{M}, (X_i, u_i, \omega_i)_{i \in N}, \Delta^{\mathcal{M}}\}$. In a usual and centralized pure exchange economy, the market structure consists of only one coalition N. Thus, our model includes the usual pure exchange model as a special case.

The decision processes for each player are as follows. First, each player will choose a combination of initial endowments for each belonging market with one's utility maximization. That is, each player plans a distribution for market A, market B, market C, etc., based on some implicit decision rule before the start of all market trades. The player will be assumed to recognize the structure of the equilibrium in the market. Second, each player will maximize one's utility among the combined budget set of the budget sets relevant to the player. Third, the equilibrium of the markets will be achieved by invisible hands. Finally, the result of the equilibrium will also influence the first decision to choose an initial endowment. All players will repeat these processes until the steady state is achieved. In steady state, the market is the Walras equilibrium, and the profile of initial endowment is also the Nash equilibrium.

2.3 The Walras Correspondence

For each economy $\mathcal{E}(\mathcal{M})$, by Assumption 2, we can define the aggregate excess demand function $Z : \Delta^{\mathcal{M}} \times \Omega \to R^l$ as follows:

$$Z(p, \Omega_1, \Omega_2, \cdots, \Omega_i, \cdots, \Omega_n) = \prod_{M \in \mathcal{M}} \left\{ \sum_{j \in M} x_j(p^M, p^M \cdot \omega_j^M) - \sum_{j \in M} \omega_j^M \right\}$$

The demand function can be defined by $x_j : \Delta^M \times \Delta^M \cdot \Omega_j^M \to X_j$ for each $j \in M \in \mathcal{M}$ as a result of each player's maximization of the player's utility u_i under the budget set $\{x_i \in X_i | x_i = \sum_{i \in M} x_i^M, p^M \cdot x_i^M \leq p^M \cdot \omega_i^M (i \in M)\}$.

Lemma 2. *The excess demand function Z is jointly continuous.*

Proof. The remarkable point of our model is that each player should evaluate their own utility by summarizing choices in the markets to which the player belongs. The utility of the player i' is redefined as $U_i = u_i \circ f_i$, $f_i : \prod_{i \in M} X_i^M \to \sum_{i \in M} X_i$. The function U_i also obviously satisfies the conditions of Assumption 2, that is, continuous, strictly quasi-concave, and strictly monotonic on $\prod_{i \in M} X_i^M$.

In light of Lemma 1, the demand correspondence $x_i(p, \mathrm{Proj}_i p \cdot \Omega_i)$ [2] is also continuous. Z is a finite product of the object that is the finite summation of the correspondence x_i minus the finite summation of ω_i. Therefore, Z is jointly continuous. *Q.E.D.*

3 An Equilibrium

The aim of this section is to prove the existence of the Walras equilibrium and the Nash-Walras equilibrium. First, we will prove the existence of a Walras equilibrium of any market structure for any initial endowment distribution.

We will introduce the concept of equilibrium for a market structure $\mathcal{M}$.

Definition 1 (A Walras equilibrium of a market structure $\mathcal{M}$ for $\omega \in \Omega$)**.** For each economy $\mathcal{E}(\mathcal{M})$, for a $\omega \in \Omega$, there exists a price $p \in \Delta^{\mathcal{M}}$, $Z(p, \Omega_1, \Omega_2, \cdots, \Omega_i, \cdots, \Omega_n) = 0$

We can also define the Walras correspondence $W : \Omega \twoheadrightarrow \Delta^{\mathcal{M}}$ as follows:

$$\omega \mapsto W(\omega) = \{p \in \Delta^{\mathcal{M}} | Z(p, \omega) = 0\}$$

The next theorem guarantees the existence of a Walras equilibrium. However, we should recognize that our formulations may be very strict and in a typical form.

Theorem 3. *For any $\mathcal{M}$, for any $\omega \in \Omega$, there exists a Walras equilibrium. That is, the Walras correspondence $W(\omega)$ is non-empty.*

Proof. By Assumption 2 and Lemma 2, we can find that our settings satisfy the standard conditions of existence for general equilibrium, for example, convexity, compactness, monotonicity, and continuity of feasible sets or utility functions. Hence, a Walras equilibrium exists. See also Mas-Colell et al. [16, Proposition 17.C.1]. *Q.E.D.*

We will express this equilibrium price for an initial endowment as $p^*(\omega) \in \Delta^{\mathcal{M}}$.

Definition 2 (A Nash-Walras equilibrium of a market structure $\mathcal{M}$)**.** There exists a Nash-Walras equilibrium of a market structure $\mathcal{M}$ if and only if

$$\forall \mathcal{M} \in \mathcal{MS}, \forall i \in N, \forall \omega_i \in \Omega_i, \exists \omega_i^* \in \Omega_i,$$
$$u_i(x_i(p^*(\omega_1^*, \omega_2^*, \cdots, \omega_n^*))) \geq u_i(x_i(p^*(\omega_i, \omega_{-i}^*))).$$

Dubey and Ruscitti [9] proved the theorem in a more general version of the following proposition.

Proposition 1 (Dubey and Ruscitti [9])**.** *Suppose that Assumptions 2 hold. Then, there exists a dense subset of Ω such that the restriction of the Walras correspondence W to this subset is continuous.*

Our proof of the next theorem is essentially dependent on Proposition 1.

[2] $\mathrm{Proj}_i p$ means the projection mapping with the price p to the player i's trading market price coordination.

Theorem 4. *For any market structure $\mathcal{M}$, there exists a Nash-Walras equilibrium.*

Proof. By Proposition 1, $\forall\omega \in \Omega, \exists D(\omega) \subset \Omega, D(\omega)$ is dense and the Walras correspondence $W(\omega)$ is continuous on $D(\omega)$. We can define the cover set as $D(\Omega) = \cup_{\omega\in\Omega} D(\omega)$. Obviously, the Walras correspondence $W(\omega)$ is also continuous on $D(\Omega)$ for all $\omega \in \Omega$. We can also confirm that $\Omega \subset D(\Omega)$. Therefore, $\forall\omega \in \Omega, W(\omega)$ is also continuous in Ω. Hence, for all $\omega \in \Omega$, for all $p^* \in W(\omega)$, we can redefine the continuous utility function u_i in Ω for $i \in N$, because the demand function x_i and the utility function u_i^* are continuous for all $p^* \in W(\omega)$. In addition, the redefined u_i^* inherits the property of the original utility function, that is, strong monotonicity and strictly quasi-concavity.

By these conditions and Lemma1, we can apply Berge's maximum theorem and Kakutani's fixed point theorem to this strategic form game. See also Berge [2, p 116, p 174] or Border [4, p 64, p 72]. This fixed point is the required equilibrium point. *Q.E.D.*

Next, Theorem 5 shows the link between the Walras equilibrium and the Nash-Walras equilibrium in our model. Busetto, Codognato, and Ghosal [5], whose model is sequential, non-cooperative, partly atomless players settings, have already shown the equivalence between the outcome of a Walras equilibrium and a subgame-perfect Nash equilibrium. Our next theorem is similar to their main theorem. Of course, our settings are different from theirs.

Theorem 5. *If a market structure $\mathcal{M}$ such that $\mathcal{M} = \{N\}$ has a Walras equilibrium, then for the market structure $\mathcal{M}' = \{\{i,j\}|\{i,j\} \subset N, i \in N, j \in N, i \neq j\}$, there exist $\omega^{*\mathcal{M}'}$, these Walras equilibrium outcomes, the equilibrium price p^* and the equilibrium demand x^*, are a Nash-Walras equilibrium.*

Proof. Let us define x^* as the offer of the Walras equilibrium, and p^* as the Walras equilibrium price. Assume that $p^* = p^{\{i,j\}}$ for all $i, j \in N, i \neq j$. If the price of all markets is equal, each merged budget set of all players is not kinked (See also Figure 3 in section 4.5, which illustrates the "kinked" budget set.). This is because the merged budget set in several pairwise markets coincides with the budget set in the centralized market. That is, for each $i \in N$,

$$\left\{x_i \in X_i \middle| x_i = \sum_{i\in M} x_i^M, p^M \cdot x_i^M \leq p^M \cdot \omega_i^M (i \in M)\right\}$$
$$=\left\{x_i \in X_i \middle| x_i = x_i, p^* \cdot x_i \leq p^* \cdot \omega_i\right\}$$
$$=\left\{x_i \in X_i \middle| x_i = \sum_{j\in N, j\neq i} x_i^{\{i,j\}}, p^* \cdot x_i^{\{i,j\}} \leq p^* \cdot \omega_i^{\{i,j\}} (j \in N, j \neq i)\right\}$$

In this case, we can freely choose the strategic initial endowment for each budget set without affecting the maximization of utilities, because the allocation plans of initial endowments do not affect the feature of their budget set. Therefore, we can concentrate on searching the balance between the strategic initial endowment and the equilibrium demand.

For each $i \in N$, we can choose x_i^*, such that $x_i^* = \sum_{j\in N, j\neq i} x_i^{*\{i,j\}}$ and $x^* = (x_1^*, \cdots, x_i^*, \cdots, x_n^*)$. Next, we should find the initial endowment $\omega^{*\mathcal{M}'}$ which satisfies the condition of each market clearing. For each $i \in N$, it should satisfy that $\omega_i = \sum_{j\neq i, j\in N} \omega_i^{*\{ij\}}$.

In addition, it also should satisfy that $\forall\{i,j\} \in \mathcal{M}'$, $\omega_i^{*\{i,j\}} + \omega_j^{*\{i,j\}} = x_i^{*\{i,j\}} + x_j^{*\{i,j\}}$. First, we will define an artificial evaluating function as

$$v_i(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}) = \sum_{j\neq i, j\in N} \sqrt{\left(\omega_i^{\{i,j\}} + \omega_j^{\{i,j\}} - x_i^{*\{i,j\}} - x_j^{*\{i,j\}}\right)^2}$$

for each $i \in N$. [3] Obviously, the function is continuous on $\Omega(\epsilon)$, which is defined in the proof of Theorem 4. We can also define the mapping as

$$V_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big) = \left\{ \bar{\omega}_i^{\mathcal{M}'} \in \Omega_i^{\mathcal{M}'} \middle| v_i\big(\bar{\omega}_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big) = \max_{\omega_i^{\mathcal{M}'} \in \Omega_i^{\mathcal{M}'}} -v_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big) \right\}$$

for each $i \in N$.

We can also apply Berge's maximum theorem to $V_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big)$, then, the mapping is upper-hemi continuous and compact valued. However, the mapping may not be convex valued. We will define the extension of the mapping as $\tilde{V}_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big) = \bar{co}V_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big)$, which is the convex hull of $V_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big)$. This mapping is also upper-hemi continuous (See Border [4, p 61]), obviously convex and compact valued. Of course, these convex combinations of the maximum points are feasible in $\Omega_i(\epsilon)$.

Next, let us construct the mapping $V\big(\omega^{\mathcal{M}'}\big)$ as $V\big(\omega^{\mathcal{M}'}\big) = \prod_{i \in N} \tilde{V}_i\big(\omega_i^{\mathcal{M}'}, \omega_{-i}^{\mathcal{M}'}\big)$. We can apply Kakutani's fixed point theorem to this mapping again. This fixed points include at least one point of the maximum of each $v_i, i \in N$. This fixed point is a candidate for the required initial endowment. We should confirm that the candidate may satisfy the market clearing conditions.

Assume that the candidate $\omega^{*\mathcal{M}}$ does not satisfy the market clearing conditions. Then, for some goods k, $\exists a \in N, \exists b \in N$, $\omega_a^{k\{a,b\}} + \omega_b^{k\{a,b\}} < x_a^{k*\{a,b\}} + x_b^{k*\{a,b\}}$. For all $M, M' \in \mathcal{M}'$ such that $M \neq M'$, $M \cap M' \neq \emptyset$, because the market structure $\mathcal{M}'$ is a complete graph in the meaning of graph theory (See Bollobás [3, p 3]), that is, all players are individually linked with all of the other players.

Hence, $\exists c \in N$, $\omega_a^{k\{a,c\}} + \omega_c^{k\{a,c\}} > x_a^{k*\{a,c\}} + x_b^{k*\{a,c\}}$, because x^* is a Walras equilibrium. For some sufficiently small $\delta > 0$, player a can transfer δ from the market $\{a, b\}$ to the market $\{a, c\}$. That is, $\omega_a^{k\{a,b\}} + \delta + \omega_b^{k\{a,b\}} < x_a^{k*\{a,b\}} + x_b^{k*\{a,b\}}$ and $\omega_a^{k\{a,c\}} - \delta + \omega_c^{k\{a,c\}} > x_a^{k*\{a,c\}} + x_b^{k*\{a,c\}}$. However, this transfer obviously improves the value of the evaluating function v_a. This is the contradiction to the fact that the ω^* is the maximum point of the fixed point of V. Therefore, the fixed point is a Nash-Walras equilibrium. *Q.E.D.*

Theorem 5 shows the robustness of the centralized market through some decentralized strategic market. Any Walras equilibrium in the centralized market can be described as a ϵ-Nash-Walras equilibrium by the appropriate allocations of the initial endowments in the pairwise markets. In other words, the pairwise economy is guaranteed to realize the outcome of the Walras equilibrium in the centralized economy. Of course, if some links between two players are cut, the coincidence of the Walras equilibrium with the Nash-Walras equilibrium is not guaranteed. These facts can be confirmed as the numerical examples in Sections 4.4 and 4.6.

4 Numerical Examples

In this section, we will show several numerical examples of our model. Several examples of the incomplete market model, including some self-consumption models, are already illustrated by Ventura[26], who provides the formulations for some self-consumption cases. We will provide other examples in our decentralized settings. In Section 4.7, we will compare the following examples and summarize the utility of the players in each case.

The common settings are as follows. $N = \{1, 2, 3\}$,$\omega_i = (3, 3) \in R_+^2$ for $i \in N$, $u_1 = \frac{1}{2}\ln x_1^1 + \frac{1}{2}\ln x_1^2$, $u_2 = \frac{8}{9}\ln x_2^1 + \frac{1}{9}\ln x_2^2$, and $u_3 = \frac{1}{9}\ln x_3^1 + \frac{8}{9}\ln x_3^2$, which are the so-called Cobb-Douglas-type utility functions.

[3] The notion of $-i$ means all players other than player i

We will present six examples of market structures in the case of three players. The following figures (Figure 2) represent coalitional situations of the six following examples. Detailed explanations are provided in the following subsections. In addition, the detailed formal settings of maximization problems, formal calculations and SymPy [4] source code are also provided in Appendix.

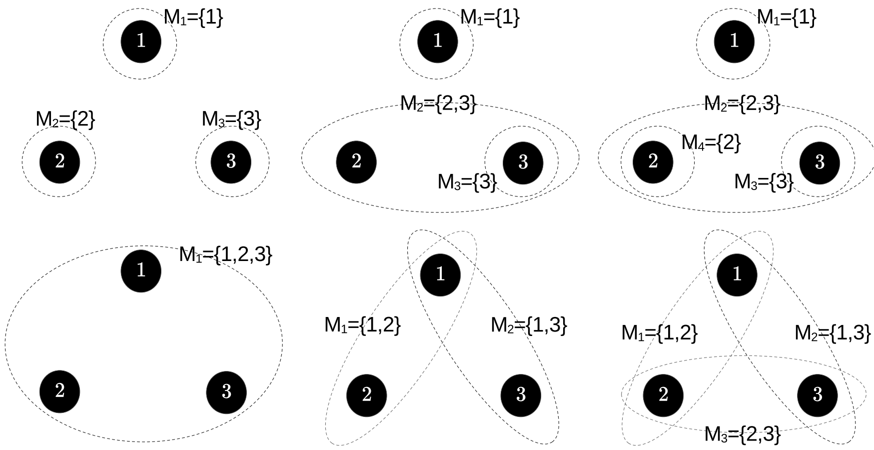


Figure 2: Six examples of market structures for three players

4.1 Robinson Crusoe economy: Case A

First, we consider the independent economy where each player is a "Robinson Crusoe" (the upper left of Figure 2). Each player may self-consume one's own initial endowment. That is, $\mathcal{M} = \{\{1\},\{2\},\{3\}\}$. All players cannot choose any combination of initial endowment for each market because they belong to only one coalition which consists of one player. They are not forced to provide their initial endowment with a public market. Therefore, each player's strategy set is a singleton set, which is a compact and convex set. That is, $\Omega_i = \{\omega_i\}$ for $i \in N$. For an equilibrium price, we can give any price that is strictly positive. In this market structure, the economy is always the Nash-Walras equilibrium for any properly imagined price. The equilibrium price is $p^{\{i\}} = p^*_{++} > 0$ for $i = 1,2,3$. Of course, their utilities are $u_1 = u_2 = u_3 = \ln(3) \fallingdotseq 1.099$.

4.2 Self-consumption vs. bilateral trade: Case B

See the upper middle of Figure 2. We will consider that player 1 still lives on a single island, player 2 trades with player 3, and player 3 also consumes a part of the initial endowment by oneself without providing one's whole initial endowment for the market of players 2 and 3. The market structure is $\mathcal{M} = \{\{1\},\{2,3\},\{3\}\}$. Player 2 will usually solve the utility maximization problem. However, player 3 will not only solve the maximization problem, but should also decide the amount of self-consumption. Because players 1 and 2 participate respectively in the single market, $\Omega_i = \{\omega_i\}$ for $i = 1,2$, which are non-strategic strategy sets. For player 3, the set of strategies for the initial endowments is $\Omega_3 = \{(\omega_3^{\{2,3\}}, \omega_3^{\{3\}}) \in R^2_+ | \omega_3^{\{2,3\}} + \omega_3^{\{3\}} = \omega_3 = (3,3)\}$. Player 3 can have monopolistic behavior. On the other hand, player 2 is a price taker. In pure

[4] SymPy(Meurer et al.[17]) is the symbolic mathematics package, which is based on the free and open source language, Python.

exchange models, many genuine articles provide the formalizations of monopolistic circumstances, including self-consumption.

Our calculation results for the Nash-Walras equilibrium are $\omega_1^{*\{1\}} = (3,3) = x_1^{*\{1\}}$,$\omega_2^{*\{2,3\}} = (3,3)$, $\omega_3^{*\{2,3\}} = (\frac{3}{2},\frac{3}{2})$,$\omega_3^{*\{3\}} = (\frac{3}{2},\frac{3}{2}) = x_3^{*\{3\}}$, $x_2^{*\{2,3\}} = (\frac{72}{17},\frac{9}{10})$, $x_3^{*\{2,3\}} = (\frac{9}{34},\frac{18}{5})$,$u_2 = \frac{8}{9}\ln(\frac{72}{17}) + \frac{1}{9}\ln(\frac{9}{10}) \fallingdotseq 1.271$, $u_3 = \frac{1}{9}\ln(\frac{30}{17}) + \frac{8}{9}\ln(\frac{51}{10}) \fallingdotseq 1.511$, $p^{\{1\}} = p^{\{3\}} = p^*_{++}$and $p^{\{2,3\}} = (\frac{17}{27},\frac{10}{27})$. Player 3, who has the monopolistic power, increases the utility compared to the previous example. Player 2 also increases the utility. However, player 2 does not reach the utility level of Walras equilibrium in the case of both players being price takers, who cannot take self-consumptions.

Player 3 obtains the monopolistic power by increasing the scarcity of the commodity to which player 3 less prefers through the strategic self-consumption decision. Good 1 is more scarce, the price of good 1 may be higher, and the feasible set of player 3 may expand. However, player 3 should also consume the less preferred good 1. Therefore, player 3 can achieve some decision-making to select the level of self-consumption.

This situation is similar to the coordination of agricultural production. For the purpose of raising the price of some crops, some farmers will self-consume or discard a part of the crops. As a result, the consumers as farmers increase the price of crops and their benefit. However, the welfare of the general consumers may decrease.

4.3 Bilateral trade with strategic initial endowment allocations: Case C

The market structure is $\mathcal{M} = \{\{1\},\{2,3\},\{3\},\{2\}\}$(The upper right of Figure 2). Player 1 still lives a wayward life alone. In addition to the previous example, player 2 is also able to reserve a part of the initial endowment for self-consumption. Both players 2 and 3 can plan strategic behavior.

The Nash-Walras equilibrium, whose concept allows a zero value of ϵ in the definition of ϵ-Nash-Walras equilibrium, is $\omega_1^{*\{1\}} = (3,3)$,$\omega_2^{*\{2\}} = (0,\frac{3}{2}) = x_2^{*\{2\}}$,$\omega_2^{*\{2,3\}} = (3,\frac{3}{2})$, $\omega_3^{*\{2,3\}} = (\frac{3}{2},3)$,$\omega_3^{*\{3\}} = (\frac{3}{2},0) = x_3^{*\{3\}}$, $x_2^{*\{2,3\}} = (4,\frac{1}{2})$, $x_3^{*\{2,3\}} = (\frac{1}{2},4)$,$u_2 = \frac{8}{9}\ln(4) + \frac{1}{9}\ln(2) = u_3 = \frac{1}{9}\ln(2) + \frac{8}{9}\ln(4) \fallingdotseq 1.309$, $p^{\{i\}} = p^*_{++}(i = 1,2,3)$ and $p^{\{2,3\}} = (\frac{1}{2},\frac{1}{2})$, which does not coincide with both the previous example and the situation in which two players are price takers.

We can also calculate the case where players 2 and 3 are price-takers, that is, $\mathcal{M} = \{\{1\},\{2,3\}\}$. The Walras equilibrium is $x_2^{*\{2,3\}} = (\frac{16}{3},\frac{2}{3})$,$x_3^{*\{2,3\}} = (\frac{2}{3},\frac{16}{3})$, $u_2 = \frac{8}{9}\ln(\frac{16}{3}) + \frac{1}{9}\ln(\frac{2}{3}) = u_3 = \frac{1}{9}\ln(\frac{2}{3}) + \frac{8}{9}\ln(\frac{16}{3}) \fallingdotseq 1.443$, $p^{\{1\}} = p^*_{++}$ and $p^{\{2,3\}} = (\frac{1}{2},\frac{1}{2})$.

The result means that the monopolistic profit of player 3 may vanish by the change of circumstances, where both player 2 and 3 are "monopolistic". That is, they are duopolistic or "Cournotian" players. Player 2 can achieve more utility than the previous example, where the other player is monopolistic. However, their utilities may decrease, compared with the gain of two players in a competitive Walrasian market. This situation coincides with the Cournot-Nash equilibrium of quantity competition in duopoly.

4.4 A centralized pure exchange economy with three players: Case D

In this subsection, we provide a numerical example of a typical centralized market to compare with the other market structures. The market structure is $\mathcal{M} = \{\{1,2,3\}\}$(The lower left of Figure 2. All players belong to one single market. In addition to the case of subsection 4.1, the set of strategies of each player is a singleton. All players are forced to provide their entire initial endowment for one single market. By the calculation of this problem, their consumption and utilities are $x_1^* = (3,3)$,$x_2^* = (\frac{16}{3},\frac{2}{3})$,$x_3^* = (\frac{2}{3},\frac{16}{3})$, $u_1 = \frac{1}{2}\ln(3) + \frac{1}{2}\ln(3) = \ln(3) \fallingdotseq 1.099$ and $u_2 = \frac{8}{9}\ln(\frac{16}{3}) + \frac{1}{9}\ln(\frac{2}{3}) = u_3 = \frac{1}{9}\ln(\frac{2}{3}) + \frac{8}{9}\ln(\frac{16}{3}) \fallingdotseq 1.443$. The (Nash-) Walras equilibrium price is $p^{\{1,2,3\}} = (\frac{1}{2},\frac{1}{2})$.

As a result of their utility maximization, player 1 does not make a trade; on the other hand, players 2 and 3 trade directly with each other. Player 1 is neutral to the preference of good 1 or 2. On the other hand, player 2 prefers good 1 to good 2, and player 3 prefers good 2 to good 1. Therefore, both player 2 and 3 need not trade with player 1. Although player 1 is a member of this centralized market, player 1 is essentially not given the role of a trader. Obviously, this centralized economy can improve the utilities of the three players in the meaning of Pareto. However, for player 1, these situations are not different from the residence on some single island.

4.5 A decentralized pure exchange economy with three players where one player is a middlemen: Case E

The market structure is $\mathcal{M} = \{\{1,2\},\{1,3\}\}$(The lower middle of Figure 2). Player 1 can trade with players 2 and 3. On the other hand, players 2 and 3 cannot directly trade with each other. If player 2 wants to trade with player 3 indirectly, player 2 should trade with player 1. In this example, player 1 may play the role of a middleman as Rubinstein and Wolinsky [20] have discussed. However, player 1 is not explicitly given the role of a middleman. The role depends on the situation of the market structure and the possibility of player 1's strategic behaviors. Player 1 may become a middleman as the result of one's own utility maximization under the circumstances of coalitional relationships. Therefore, our model may implicitly include the situation of middlemen.

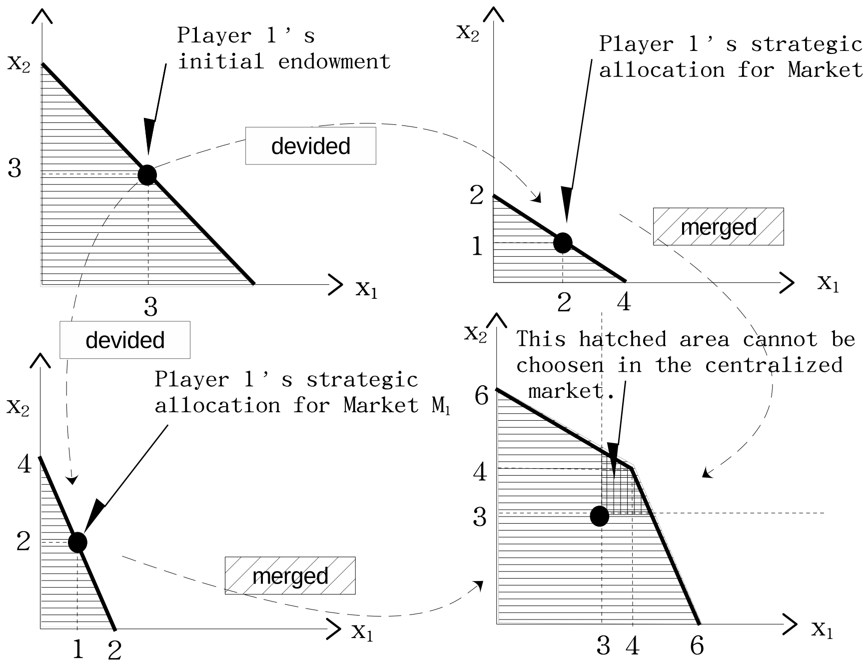


Figure 3: A budget set in a decentralized market

Figure 4.5 describes the budget set of player 1. The graph in the upper left corner is the initial budget set of player 1 before dividing the initial endowments to the market M_1 and M_2. The budget set is realized if the price $p^{\{1,2\}} = p^{\{1,3\}} = (\frac{1}{2}, \frac{1}{2})$. Player 1 will plan to divide the initial endowments $\omega_1 = (3,3)$ where $\omega_1^{\{1,2\}} = (1,2)$ for the market $\{1,2\}$ and $\omega_1^{\{1,3\}} = (2,1)$. The graph in the lower left corner shows the set of budgets for the market $\{1,2\}$ and the upper right corner shows the market $\{1,3\}$'s. In the lower right corner, these budget sets are merged. The feasible set of player 1 is kinked and extended, compared to the feasible set of the upper left corner. At this kink point, player 1 will maximize the utility in this set of prices of $p^{\{1,2\}} = (\frac{2}{3}, \frac{1}{3})$ and $p^{\{1,3\}} = (\frac{1}{3}, \frac{2}{3})$. The offer for the market $\{1,2\}$ is $x_1^{\{1,2\}} = (0,4)$. The other offer for the market $\{1,3\}$ is $x_1^{\{1,3\}} = (0,4)$. Hence, the total offer of player 1 is $x_1 = (0,4) + (4,0) = (4,4)$, which is never realized in the central economy. Players 2 and 3 may also maximize their utilities, where the demand for player 2 is $x_2 = (4,1)$ and the demand for player 3 is $x_3 = (1,4)$. These are the Walras equilibrium for the market structure $\mathcal{M} = \{\{1,2\},\{1,3\}\}$ and the initial endowment allocations $\omega_1^{\{1,2\}} = (1,2)$,$\omega_1^{\{1,3\}} = (2,1)$, $\omega_2^{\{1,2\}} = (3,3) = \omega_2$, and $\omega_3^{\{1,3\}} = (3,3) = \omega_3$. And their utilities are $u_1^* = \frac{1}{2}\ln(4) + \frac{1}{2}\ln(4) \risingdotseq 1.386$ and $u_2^* = u_3^* = \frac{8}{9}\ln(4) + \frac{1}{9}\ln(1) = 1.232$.

Because player 1, as the middleman, may be monopolistic, the player can improve one's own utility by selecting other combinations of initial endowment allocations. Player 1 sells a small amount of good 1, which is more preferred by player 2. Then, player 1 can buy a large amount of good 2, which is less preferred by player 2. Player 1 for player 3 is the same as for player 2. Player 1 can receive many benefits by playing the role of a "broker".

However, the Walras equilibrium is not a Nash-Walras equilibrium. Player 1 can improve one's own utility by more accurately calculating the initial strategic endowment allocations. On the other hand, players 2 and 3 may decrease their utilities.

Finally, we will compute the (monopolistic) Nash-Walras equilibrium of the market structure. The Nash-Walras equilibrium is $p^{\{1,2\}} = (\frac{2\sqrt{2}}{2\sqrt{2}+1}, \frac{1}{2\sqrt{2}+1})$, $p^{\{1,3\}} = (\frac{1}{2\sqrt{2}+1}, \frac{2\sqrt{2}}{2\sqrt{2}+1})$, $x_1^* = (x_1^{1*}, x_1^{2*}) = (x_1^{1\{1,2\}} + x_1^{1\{1,3\}}, x_1^{2\{1,2\}} + x_1^{2\{1,3\}}) = ((0 + (6 - \frac{4\sqrt{2}}{3})),((6 - \frac{4\sqrt{2}}{3}) + 0) = (6 - \frac{4\sqrt{2}}{3}, 6 - \frac{4\sqrt{2}}{3}) \risingdotseq (4.114, 4.114)$, $\omega_1^{*\{1,2\}} = (-\frac{2}{3} + \frac{2\sqrt{2}}{3}, \frac{10}{3} - \frac{2\sqrt{2}}{3}) \risingdotseq (0.609, 2.391)$, $\omega_1^{*\{1,3\}} = (\frac{10}{3} - \frac{2\sqrt{2}}{3}, -\frac{2}{3} + \frac{2\sqrt{2}}{3}) \risingdotseq (2.391, 0.609)$, $u_1^*(x_1^*) = \frac{1}{2}\ln(-\frac{4\sqrt{2}}{3} + 6) + \frac{1}{2}\ln(-\frac{4\sqrt{2}}{3} + 6) \risingdotseq 1.414$ and $u_2^*(x_2^*) = u_3^*(x_3^*) = \frac{8}{9}\ln(\frac{2\sqrt{2}}{3} + \frac{8}{3}) + \frac{1}{8}\ln(\frac{2\sqrt{2}}{3} + \frac{1}{3}) \risingdotseq 1.168$. These are the Nash-Walras equilibrium, which refines the above Walras equilibrium. Therefore, we can interpret that the Nash-Walras equilibrium may be some refinement of the Walras equilibrium by the strategic concept of Nash equilibrium.

4.6 A decentralized pure exchange economy with three players where all players poses the strategy set: Case F

The last example is the case in which all players can strategically and potentially choose their own initial endowments for each market. Exactly as player 1 may behave in the previous example, players 2 and 3 will also plan the choice of ω_2 and ω_3 for the market structure $\mathcal{M} = \{\{1,2\},\{2,3\},\{1,3\}\}$.

However, all players cannot simultaneously exercise strategic behavior in this market. If the price of each market is different from the others as in Case E, the maximum of their utilities is the point of the kinked budget set, as shown in Figure 3. If all players select the point, we should accept the utopia world. If the prices of all markets are equivalent, the allocations of the initial endowment are indifference. We can confirm that the strategic behavior of the players is meaningless. Therefore, the result of the Nash-Walras equilibrium coincides with the case of section 4.4. In addition, we can find the appropriate allocations of the initial endowment following this equilibrium. We can also find that the role of player 1 as the middleman may vanish. The result of this numerical example is consistent with Theorem 5.

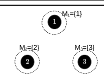
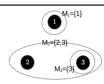
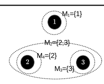
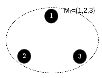
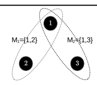
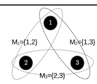
	u_1	u_2	u_3	$\mathcal{M}$	
Case A	1.099	1.099	1.099	$\{1\}, \{2\}, \{3\}$	
Case B	1.099	1.271	1.511	$\{1\}, \{2, 3\}, \{3\}$	
Case C	1.099	1.309	1.309	$\{1\}, \{2\}, \{2, 3\}, \{3\}$	
Case D	1.099	1.443	1.443	$\{1, 2, 3\}$	
Case E	1.414	1.168	1.168	$\{1, 2\}, \{1, 3\}$	
Case F	1.099	1.443	1.443	$\{1, 2\}, \{2, 3\}, \{1, 3\}$	

Table 1: Utilities of 3 players

4.7 A summary of examples

We will show the summary of the above examples in Table 1.

Obviously, their utilities of Cases B, C, D, E, and F are more efficient than those of Case A. All players may be eager to deviate from the situation of Robinson Crusoe. If players trade in Case C, they can increase their utilities by making a coalition $\{1, 2, 3\}$, that is, the centralized market Case D. However, other cases are not comparable with each other.

For example, although the most desirable target of player 1 may be Case E, the other player does not want the case. For the deviation of player 1 from the level of initial endowment, player 1 should divide the linkage between player 2 and 3. For peacefully increasing the utilities of players 2 and 3, they should keep their coalition sustained. In Case F, all players are linked to the other players. Then, the benefit of player 1 may be extinguished again.

Therefore, each case may not be a stable state in the meaning of the formation about market structures. Of course, it is important that we become able to compare among different coalitional situations on that common platform.

5 Concluding Remarks

Our model may succeed in describing the situation in which the market structure decides the market allocation in the steady state. As shown in Section 4, the diversity of the utilities of the players in the Nash-Walras equilibrium may depend on the structure of the market or the coalitional characteristic, even if all their initial endowments are equal. In our model, the role of each player may also be decided by the market structure, even if the behavioral rule is common. Someone is a Robinson Crusoe. Someone is a middleman. Someone is a monopolistic trader. The others are Curnotish, Nashian, Walrasian or other players who make complex behaviors.

These different roles are not explicitly given. Each player will behave according to only two individual disciplines. One is to maximize the utility given for the price and the initial endowment given under budget constraints. The other is to maximize utility given the initial strategic endowment allocations of other players and the market structure. However, each consumer may achieve a task as a non-cooperative player without receiving any scheme for controlling the market.

In our model, we can also confirm that the key factor of the consequence in the market is the characteristic of the market structure or the structure of coalitions including the network models. We believe that this descriptive possibility may be the main point of our article's advantage over other models of the decentralized market.

Of course, it may be difficult to calculate the equilibria of more huge economic models compared to our numerical examples. Therefore, for the application of our model, we need to simplify the utility functions or other settings.

In this article, we also provide the equilibrium concept, that is, the Nash-Walras equilibrium. The concept may be naturally introduced to our model. We can also recognize this equilibrium concept as the refinement of Walras equilibria in our model. In other words, we may rationalize the Walras equilibria by the concept of Nash equilibrium. Therefore, in future studies, we can replace the Nash equilibrium with some solution concepts of the strategic form game , that is, strong Nash equilibrium, α-core, β-core, etc., although we need to introduce the other coalitional concepts in our coalitional model. We believe in the possibility that our model may expand the descriptive or positive area of the coalitional and strategic economy.

A Calculations of Numerical Examples

For each market, the price of good 1 is normalized by the price of good 2. Therefore, in the following calculation process, the price of good 2 is always 1. The normalized price of good 1, that is p^1p^2, is re-defined by p^1.

A.1 Case B in Section 4.2

Let us construct the utility maximization problem for each player. For player 1, we do not need any construction. For player 2,

$$\begin{array}{cl} \max_{x_2^1, x_2^2} & u_2 = \frac{8}{9}\ln x_2^1 + \frac{1}{9}\ln x_2^2 \\ \text{s.t.} & p^{1\{2,3\}} \cdot x_2^1 + 1 \cdot x_2^2 \leq p^{1\{2,3\}} \cdot 3 + 1 \cdot 3 \end{array}$$

The lagrangian of this maximization problem is

$$\mathcal{L}_2 = u_2 + \lambda_2(p^{1\{2,3\}} \cdot x_2^1 + x_2^2 - 3p^{1\{2,3\}} - 3)$$

For player 3,

$$\begin{array}{cl} \max_{x_3^1, x_3^2} & u_3 = \frac{1}{9}\ln x_3^1 + \frac{8}{9}\ln x_3^2 \\ \text{s.t.} & p^{1\{2,3\}} \cdot +x_3^1 + 1 \cdot x_3^2 \leq p^{1\{2,3\}} \cdot (\omega_3^{1\{2,3\}} + \omega_3^{1\{3\}}) + 1 \cdot (\omega_3^{2\{2,3\}} + \omega_3^{2\{3\}}) \\ & \omega_3^{1\{2,3\}} + \omega_3^{1\{3\}} = 3 \\ & \omega_3^{2\{2,3\}} + \omega_3^{2\{3\}} = 3 \end{array}$$

The lagrangian of the player 3's maximization problem is

$$\mathcal{L}_3 = u_3 + \lambda_3(p^{1\{2,3\}} \cdot x_3^1 + x_3^2 - p^{1\{2,3\}}(3 - \omega_3^{1\{3\}}) - \omega_3^{2\{3\}})$$

The first order conditions are the followings; $\partial\mathcal{L}_2/\partial x_2^1 = 0$, $\partial\mathcal{L}_2/\partial x_2^2 = 0$, $\partial\mathcal{L}_2/\partial\lambda_2 = 0$, $\partial\mathcal{L}_3/\partial x_3^1 = 0$, $\partial\mathcal{L}_3/\partial x_3^2 = 0$,and $\partial\mathcal{L}_3/\partial\lambda_3 = 0$. We can solve these equations. We have $x_2^1 = 8(p^{1\{2,3\}} + 1)/3p^{1\{2,3\}}$, $x_2^2 = p^{1\{2,3\}}/3 + 1/3$, and $\lambda_2 = -1/(3p^{1\{2,3\}} + 3)$. Also, we have

$$\begin{array}{rl} x_3^1 = & (-\omega_3^{1\{3\}}p^{1\{2,3\}} - \omega_3^{2\{3\}} + 3p^{1\{2,3\}} + 3)/9p^{1\{2,3\}}, \\ x_3^2 = & -8\omega_3^{1\{3\}}p^{1\{2,3\}}/9 - 8\omega_3^{2\{3\}}/9 + 8p^{1\{2,3\}} + 8/3, and \\ \lambda_3 = & 1/(\omega_3^{1\{3\}}p^{1\{2,3\}} + \omega_3^{2\{3\}} - 3p^{1\{2,3\}} - 3). \end{array} \quad (1)$$

The conditions of market equilibrium are $x_2^1+x_3^1=3+(3-\omega_3^{1\{3\}})$ and $x_2^2+x_3^2=3+(3-\omega_3^{2\{3\}})$.

We can substitute x_2^1, x_3^1, x_2^2, and x_3^2 for these equations. We can solve these equations and we have the price,

(2) $\quad p^{1\{2,3\}}=(\omega_3^{2\{3\}}-27)/(8\omega_3^{1\{3\}}-27)$.

Next, we should calculate the maximum by initial endowment allocations. We will replace the demand (x_3^1,x_3^2) of (1) solved by the maximization problem with $(x_3^1+\omega_3^{1\{3\}},x_3^2+\omega_3^{2\{3\}})$ including an initial endowment plan for the market $\{3\}$, which is consumed by oneself. We will also substitute this newly defined demand, the allocation of strategic initial endowment, and the solved price (2) for the utility functions of player 2 and 3. The utility function of player 2 in the market equilibrium is

$$u_2=\frac{8\ln\left(\frac{8\left(1+\frac{\omega_3^{2\{3\}}-27}{8\omega_3^{1\{3\}}-27}\right)\left(8\omega_3^{1\{3\}}-27\right)}{3\left(\omega_3^{2\{3\}}-27\right)}\right)}{9}+\frac{\ln\left(\frac{1}{3}+\frac{\omega_3^{2\{3\}}-27}{3\left(8\omega_3^{1\{3\}}-27\right)}\right)}{9}$$

The utility function of player 3 in the market equilibrium is

$$u_3=\frac{\ln\left(\omega_3^{1\{3\}}+\frac{\left(8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}\right)}{9\left(\omega_3^{2\{3\}}-27\right)}\right)}{9}+\frac{8\ln\left(-\frac{8\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{9\left(8\omega_3^{1\{3\}}-27\right)}+\frac{\omega_3^{2\{3\}}}{9}+\frac{8}{3}+\frac{8\left(\omega_3^{2\{3\}}-27\right)}{3\left(8\omega_3^{1\{3\}}-27\right)}\right)}{9}$$

The first order condition of the maximization choosing the strategies, that is, the allocations of initial endowments, we will differentiate these utility functions. We can obtain simultaneous equations below.

$$\frac{\partial u_3}{\partial\omega_3^{1\{3\}}}=\frac{8\left(\frac{64\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{9\left(8\omega_3^{1\{3\}}-27\right)^2}-\frac{8\left(\omega_3^{2\{3\}}-27\right)}{9\left(8\omega_3^{1\{3\}}-27\right)}-\frac{64\left(\omega_3^{2\{3\}}-27\right)}{3\left(8\omega_3^{1\{3\}}-27\right)^2}\right)}{9\left(-\frac{8\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{9\left(8\omega_3^{1\{3\}}-27\right)}+\frac{\omega_3^{2\{3\}}}{9}+\frac{8}{3}+\frac{8\left(\omega_3^{2\{3\}}-27\right)}{3\left(8\omega_3^{1\{3\}}-27\right)}\right)}+\frac{\frac{\left(8\omega_3^{1\{3\}}-27\right)\left(\frac{8\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{\left(8\omega_3^{1\{3\}}-27\right)^2}-\frac{\omega_3^{2\{3\}}-27}{8\omega_3^{1\{3\}}-27}-\frac{24\left(\omega_3^{2\{3\}}-27\right)}{\left(8\omega_3^{1\{3\}}-27\right)^2}\right)}{9\left(\omega_3^{2\{3\}}-27\right)}+1+\frac{8\left(-\frac{\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}\right)}{9\left(\omega_3^{2\{3\}}-27\right)}}{9\left(\omega_3^{1\{3\}}+\frac{\left(8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}\right)}{9\left(\omega_3^{2\{3\}}-27\right)}\right)}=0$$

$$\frac{\partial u_3}{\partial\omega_3^{2\{3\}}}=\frac{8\left(-\frac{8\omega_3^{1\{3\}}}{9\left(8\omega_3^{1\{3\}}-27\right)}+\frac{1}{9}+\frac{8}{3\left(8\omega_3^{1\{3\}}-27\right)}\right)}{9\left(-\frac{8\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{9\left(8\omega_3^{1\{3\}}-27\right)}+\frac{\omega_3^{2\{3\}}}{9}+\frac{8}{3}+\frac{8\left(\omega_3^{2\{3\}}-27\right)}{3\left(8\omega_3^{1\{3\}}-27\right)}\right)}+\frac{\frac{\left(8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}}{8\omega_3^{1\{3\}}-27}-1+\frac{3}{8\omega_3^{1\{3\}}-27}\right)}{9\left(\omega_3^{2\{3\}}-27\right)}-\frac{\left(8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}\right)}{9\left(\omega_3^{2\{3\}}-27\right)^2}}{9\left(\omega_3^{1\{3\}}+\frac{\left(8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(\omega_3^{2\{3\}}-27\right)}{8\omega_3^{1\{3\}}-27}\right)}{9\left(\omega_3^{2\{3\}}-27\right)}\right)}=0$$

The system of equations above is very complex. We can hardly solve it directly. We should look for the focal point of the maximization problem. Let us plot the graph of utility functions with the fixation of one parameter of the initial endowment. We will substitute $\frac{3}{2}$ for $\omega_3^{2\{3\}}$ of u_3. Then

$$u_3(\omega_3^{1\{3\}}) = \frac{\ln\left(\omega_3^{1\{3\}} - \frac{2\left(8\omega_3^{1\{3\}} - 27\right)\left(\frac{51\omega_3^{1\{3\}}}{2\left(8\omega_3^{1\{3\}} - 27\right)} + \frac{3}{2} - \frac{153}{2\left(8\omega_3^{1\{3\}} - 27\right)}\right)}{459}\right)}{9} + \frac{8\ln\left(\frac{68\omega_3^{1\{3\}}}{3\left(8\omega_3^{1\{3\}} - 27\right)} + \frac{17}{6} - \frac{68}{8\omega_3^{1\{3\}} - 27}\right)}{9}$$

We can plot the graph of $u_3(\omega_3^{1\{3\}})$. We can expect a point around the $\omega_3^{1\{3\}} = 3/2$ maximum.

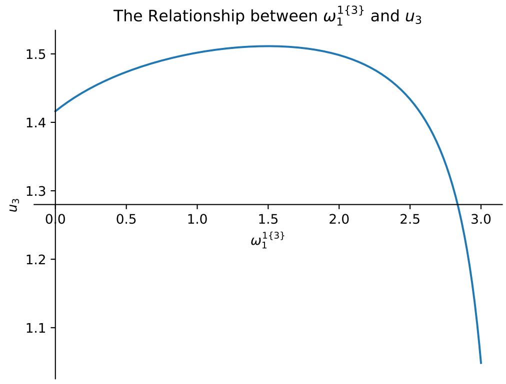


Figure 4: The relationship between initial endowments $\omega_3^{1\{3\}}$ and the utility of player 3

We will also substitute $\frac{3}{2}$ for $\omega_3^{2\{3\}}$ of u_3. Then, we have

$$u_3(\omega_3^{2\{3\}}) = \frac{8\ln\left(\frac{\omega_1^{2\{3\}}}{45} + \frac{76}{15}\right)}{9} + \frac{\ln\left(-\frac{5\left(\frac{57}{10} - \frac{11\omega_1^{2\{3\}}}{10}\right)}{3\left(\omega_1^{2\{3\}} - 27\right)} + \frac{3}{2}\right)}{9}$$

We can also plot the graph of $u_3(\omega_3^{2\{3\}})$.

We can also expect a point around $\omega_3^{1\{3\}} = 3/2$ maximum.

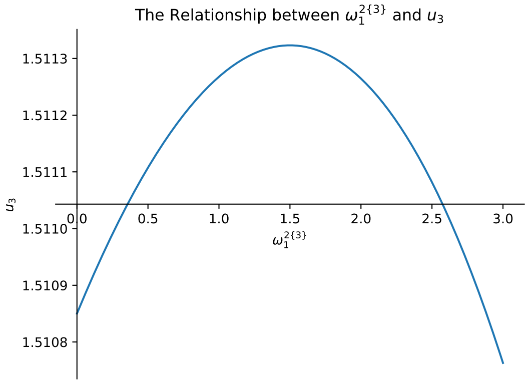


Figure 5: The relationship between the initial endowment $\omega_3^{2\{3\}}$ and the utility of player 3

Finally, we should confirm these expectations that the utility of player 3 is maximum at the points $\omega_3^{1\{3\}} = 3/2$ and $\omega_3^{2\{3\}} = 3/2$. We will substitute $\omega_3^{1\{3\}} = 3/2$ for $\frac{\partial u_3}{\partial \omega_3^{1\{3\}}}$ and $\frac{\partial u_3}{\partial \omega_3^{2\{3\}}}$, and we will solve these equations about $\omega_3^{2\{3\}}$.

Then, the solution is $\omega_3^{2\{3\}} = 3/2$. Therefore, the strategic allocation of the initial endowments $(\omega_3^{1\{3\}}, \omega_3^{2\{3\}}) = (3/2, 3/2)$ and $(\omega_3^{1\{2,3\}}, \omega_3^{2\{2,3\}}) = (3/2, 3/2)$ maximize the utility function of player 3. Hence, this is the Nash-Walras equilibrium.

We can substitute the above results for the equilibrium price expressed by the initial endowment variables (2) and the first order conditions (1). Then, we can obtain the required results.

A.2 Case C in Section 4.3

Let us construct the utility maximization problem for each player. For player 1, we do not need any construction also. For player 2,

$$\begin{array}{ll} \max_{x_2^1, x_2^2} & u_2 = \frac{8}{9}\ln x_2^1 + \frac{1}{9}\ln x_2^2 \\ \text{s.t.} & p^{1\{2,3\}} \cdot x_2^1 + 1 \cdot x_2^2 \leq p^{1\{2,3\}} \cdot (\omega_2^{1\{2,3\}} + \omega_2^{1\{2\}}) + 1 \cdot (\omega_2^{2\{2,3\}} + \omega_2^{2\{2\}}) \\ & \omega_2^{1\{2\}} + \omega_2^{1\{2,3\}} = 3 \\ & \omega_2^{2\{2\}} + \omega_2^{2\{2,3\}} = 3 \end{array}$$

The lagrangian of this maximization problem is

$$\mathcal{L}_2 = u_2 + \lambda_2(p^{1\{2,3\}} \cdot x_2^1 + x_2^2 - p^{1\{2,3\}}(3 - \omega_2^{1\{3\}}) - \omega_2^{2\{3\}})$$

For player 3,

$$\begin{array}{ll} \max_{x_3^1, x_3^2} & u_3 = \frac{1}{9}\ln x_3^1 + \frac{8}{9}\ln x_3^2 \\ \text{s.t.} & p^{1\{2,3\}} \cdot + x_3^1 + 1 \cdot x_3^2 \leq p^{1\{2,3\}} \cdot (\omega_3^{1\{2,3\}} + \omega_3^{1\{3\}}) + 1 \cdot (\omega_3^{2\{2,3\}} + \omega_3^{2\{3\}}) \\ & \omega_3^{1\{2,3\}} + \omega_3^{1\{3\}} = 3 \\ & \omega_3^{2\{2,3\}} + \omega_2^{2\{3\}} = 3 \end{array}$$

The lagrangian of the player 3's maximization problem is

$$\mathcal{L}_3 = u_3 + \lambda_3(p^{1\{2,3\}} \cdot x_3^1 + x_3^2 - p^{1\{2,3\}}(3 - \omega_3^{1\{3\}}) - \omega_3^{2\{3\}})$$

The first order conditions are the followings; $\partial\mathcal{L}_2/\partial x_2^1 = 0$, $\partial\mathcal{L}_2/\partial x_2^2 = 0$, $\partial\mathcal{L}_2/\partial\lambda_2 = 0$, $\partial\mathcal{L}_3/\partial x_3^1 = 0$, $\partial\mathcal{L}_3/\partial x_3^2 = 0$,and $\partial\mathcal{L}_3/\partial\lambda_3 = 0$. We can solve these equations. We have

$$(3)\qquad \begin{aligned} x_2^1 &= 8(-\omega_2^{1\{2\}}p^{1\{2,3\}} - \omega_2^{2\{2\}} + 3p^{1\{2,3\}} + 3)/9p^{1\{2,3\}}, \\ x_2^2 &= -\omega_2^{1\{2\}}p^{1\{2,3\}}/9 - \omega_2^{2\{2\}}/9 + p^{1\{2,3\}}/3 + 1/3, and \\ \lambda_2 &= 1/(\omega_2^{1\{2\}}p^{1\{2,3\}} + \omega_2^{2\{2\}} - 3p^{1\{2,3\}} - 3). \end{aligned}$$

Also, we have

$$(4)\qquad \begin{aligned} x_3^1 &= (-\omega_3^{1\{3\}}p^{1\{2,3\}} - \omega_3^{2\{3\}} + 3p^{1\{2,3\}} + 3)/9p^{1\{2,3\}}, \\ x_3^2 &= -8\omega_3^{1\{3\}}p^{1\{2,3\}}/9 - 8\omega_3^{2\{3\}}/9 + 8p^{1\{2,3\}}/3 + 8/3, and \\ \lambda_3 &= 1/(\omega_3^{1\{3\}}p^{1\{2,3\}} + \omega_3^{2\{3\}} - 3p^{1\{2,3\}} - 3). \end{aligned}$$

The conditions of market equilibrium are $x_2^1 + x_3^1 = (3 - \omega_2^{1\{2\}}) + (3 - \omega_3^{1\{3\}})$ and $x_2^2 + x_3^2 = (3 - \omega_2^{2\{2\}}) + (3 - \omega_3^{2\{3\}})$.

We can substitute x_2^1, x_3^1, x_2^2, and x_3^2 for these equations. We can solve these equations and we have the price,

$$(5)\qquad p^{1\{2,3\}} = p^{2\{2,3\}} = (8\omega_2^{2\{2\}} + \omega_3^{2\{3\}} - 27)/(\omega_2^{1\{2\}} + 8\omega_3^{1\{3\}} - 27).$$

Next, we should calculate the maximum by initial endowment allocations. Similarly as in the previous section, we will replace the demand (x_2^1, x_2^2) of (3) solved by the maximization problem with $(x_2^1 + \omega_2^{1\{2\}}, x_2^2 + \omega_2^{2\{2\}})$ including an initial endowment plan for the market $\{2\}$, which is consumed by oneself. And we will also replace the demand (x_3^1, x_3^2) of (4) with $(x_3^1 + \omega_3^{1\{3\}}, x_3^2 + \omega_3^{2\{3\}})$.

We will substitute the newly replaced demands, the allocations of strategic initial endowments, and the solved price for the utility functions of player 2 and 3. The utility function of player 2 is .

$$u_2 = \frac{8\ln\left(\omega_2^{1\{2\}} + \frac{8\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27} - \omega_2^{2\{2\}} + 3 + \frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)}{9} + \frac{\ln\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27\right)} + \frac{8\omega_2^{2\{2\}}}{9} + \frac{1}{3} + \frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27\right)}\right)}{9}.$$

The utility function of player 3 is

$$u_3 = \frac{\ln\left(\omega_3^{1\{2\}} + \frac{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27\right)\left(-\frac{\omega_3^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27} - \omega_3^{2\{3\}} + 3 + \frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)}{9} + \frac{8\ln\left(-\frac{8\omega_3^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27\right)} + \frac{\omega_3^{2\{3\}}}{9} + \frac{8}{3} + \frac{8\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{2\}}-27\right)}\right)}{9}.$$

As we investigated in the previous section, we should look for the focal point of the maximization problem. Let us also plot the graph of utility functions with the fixation of one parameter of the initial endowment. We will substitute $\frac{3}{2}$ for $\omega_2^{2\{2\}}$ and $\omega_3^{1\{3\}}$, and substitute 0 for $\omega_2^{1\{2\}}$ and $\omega_3^{2\{3\}}$. Let us fix three of four parameters; $\omega_2^{1\{2\}}$, $\omega_2^{2\{2\}}$, $\omega_3^{1\{3\}}$ and $\omega_3^{2\{3\}}$. We will consider the graph of the utility function represented by one parameter of the initial endowments.

$$u_2\Big|_{\omega_2^{2\{2\}}=\frac{3}{2},\omega_3^{1\{3\}}=\frac{3}{2},\omega_3^{2\{3\}}=0} = \frac{8\ln\left(\omega_2^{1\{2\}} - \frac{8\left(\omega_2^{1\{2\}}-15\right)\left(\frac{15\omega_2^{1\{2\}}}{\omega_2^{1\{2\}}-15}+\frac{3}{2}-\frac{45}{\omega_2^{1\{2\}}-15}\right)}{135}\right)}{9} + \frac{\ln\left(\frac{5\omega_2^{1\{2\}}}{3\left(\omega_2^{1\{2\}}-15\right)}+\frac{5}{3}-\frac{5}{\omega_2^{1\{2\}}-15}\right)}{9}$$

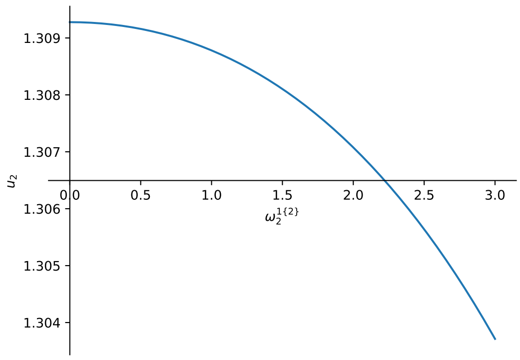


Figure 6: The relationship between initial endowments $\omega_2^{1\{2\}}$ and the utility of player 2

$$u_2\Big|_{\omega_2^{1\{2\}}=0,\omega_3^{1\{3\}}=\frac{3}{2},\omega_3^{2\{3\}}=0} = \frac{8\ln\left(-\frac{40\left(\frac{42}{5}-\frac{13\omega_2^{2\{2\}}}{5}\right)}{3\left(8\omega_2^{2\{2\}}-27\right)}\right)}{9} + \frac{\ln\left(\frac{32\omega_2^{2\{2\}}}{45}+\frac{14}{15}\right)}{9}$$

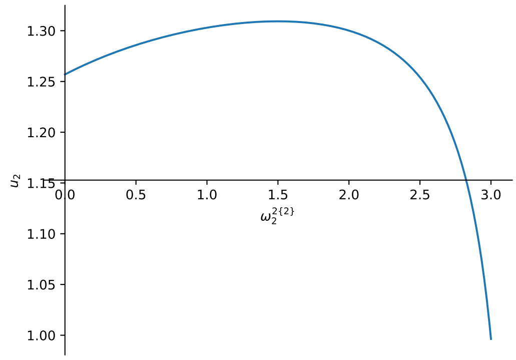


Figure 7: The relationship between initial endowments $\omega_2^{2\{2\}}$ and the utility of player 2

$$u_3\Big|_{\omega_2^{1\{2\}}=0,\omega_2^{2\{2\}}=\frac{3}{2},\omega_3^{2\{3\}}=0} = \frac{\ln\left(\omega_3^{1\{3\}} - \frac{\left(8\omega_3^{1\{3\}}-27\right)\left(\frac{15\omega_3^{1\{3\}}}{8\omega_3^{1\{3\}}-27}+3-\frac{45}{8\omega_3^{1\{3\}}-27}\right)}{135}\right)}{9} + \frac{8\ln\left(\frac{40\omega_3^{1\{3\}}}{3\left(8\omega_3^{1\{3\}}-27\right)}+\frac{8}{3}-\frac{40}{8\omega_3^{1\{3\}}-27}\right)}{9}$$

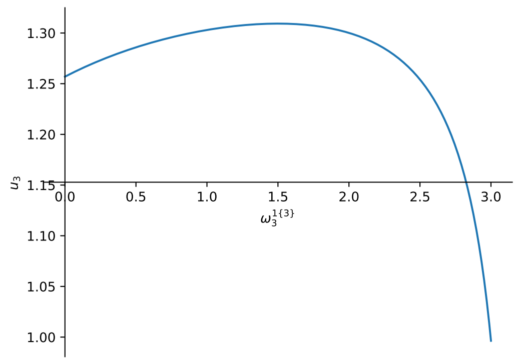


Figure 8: The relationship between initial endowments $\omega_3^{1\{3\}}$ and the utility of player 3

$$u_3\Big|_{\omega_2^{1\{2\}}=0,\omega_2^{2\{2\}}=\frac{3}{2},\omega_3^{1\{3\}}=\frac{3}{2}} = \frac{8\ln\left(\frac{\omega_3^{2\{3\}}}{45}+4\right)}{9} + \frac{\ln\left(-\frac{5\left(\frac{9}{2}-\frac{11\omega_3^{2\{3\}}}{10}\right)}{3\left(\omega_3^{2\{3\}}-15\right)}+\frac{3}{2}\right)}{9}$$

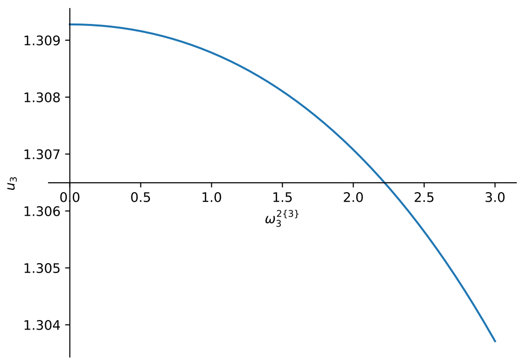


Figure 9: The relationship between initial endowments $\omega_3^{2\{3\}}$ and the utility of player 3

As we consider these four plotting graphs, the focal points may be around the places where $\omega_2^{1\{2\}} = 0$, $\omega_2^{2\{2\}} = 3/2$, $\omega_3^{1\{3\}} = 3/2$, and $\omega_3^{2\{3\}} = 0$.

Finally, we should confirm the expectations that the utility of player 3 is maximum at these points. The first order condition of the maximization choosing the strategies, that is, the allocations of initial endowments, we will differentiate these utility functions. We can also obtain simultaneous equations below.

$$\frac{\partial u_2}{\partial \omega_2^{1\{2\}}} = \frac{\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2} - \frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)} - \frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2}}{9\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)} + \frac{8\omega_2^{2\{2\}}}{9} + \frac{1}{3} + \frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}\right)}$$

$$+ \frac{8\left(\frac{8\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2} - \frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27} - \frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)} + 1 + \frac{8\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27} - \omega_2^{2\{2\}} + 3 + \frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)}{9\left(\omega_2^{1\{2\}} + \frac{8\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27} - \omega_2^{2\{2\}} + 3 + \frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)} = 0$$

$$\frac{\partial u_2}{\partial \omega_2^{2\{2\}}} = \frac{-\frac{8\omega_2^{1\{2\}}}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}+\frac{8}{9}+\frac{8}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}}{9\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}+\frac{8\omega_2^{2\{2\}}}{9}+\frac{1}{3}+\frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}\right)} + \frac{8\left(\frac{8\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{8\omega_2^{1\{2\}}}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-1+\frac{24}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)} - \frac{64\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\omega_2^{2\{2\}}+3+\frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)^2}\right)}{9\left(\omega_2^{1\{2\}}+\frac{8\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_2^{1\{2\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\omega_2^{2\{2\}}+3+\frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)} = 0$$

$$\frac{\partial u_3}{\partial \omega_3^{1\{3\}}} = \frac{8\left(\frac{64\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2}-\frac{8\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}-\frac{64\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2}\right)}{9\left(-\frac{8\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}+\frac{\omega_3^{2\{3\}}}{9}+\frac{8}{3}+\frac{8\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}\right)} + \frac{\frac{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(\frac{8\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2}-\frac{8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\frac{24\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)^2}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)} + 1 + \frac{8\left(-\frac{\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}}{9\left(\omega_3^{1\{3\}}+\frac{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)} = 0$$

$$\frac{\partial u_3}{\partial \omega_3^{2\{3\}}} = \frac{8\left(-\frac{8\omega_3^{1\{3\}}}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}+\frac{1}{9}+\frac{8}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}\right)}{9\left(-\frac{8\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{9\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}+\frac{\omega_3^{2\{3\}}}{9}+\frac{8}{3}+\frac{8\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{3\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)}\right)} + \frac{\frac{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-1+\frac{3}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)} - \frac{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)^2}}{9\left(\omega_3^{1\{3\}}+\frac{\left(\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27\right)\left(-\frac{\omega_3^{1\{3\}}\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}-\omega_3^{2\{3\}}+3+\frac{3\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}{\omega_2^{1\{2\}}+8\omega_3^{1\{3\}}-27}\right)}{9\left(8\omega_2^{2\{2\}}+\omega_3^{2\{3\}}-27\right)}\right)} = 0$$

We will substitute both $\omega_2^{1\{2\}} = 0$ and $\omega_3^{2\{3\}} = 0$ for $\frac{\partial u_2}{\partial \omega_2^{1\{2\}}}$, $\frac{\partial u_2}{\partial \omega_2^{2\{2\}}}$, $\frac{\partial u_3}{\partial \omega_3^{1\{3\}}}$, and $\frac{\partial u_3}{\partial \omega_3^{2\{3\}}}$, and we will solve these equations about $\omega_2^{2\{2\}}$ and $\omega_3^{1\{3\}}$.

Then, the solution is $\omega_2^{2\{2\}} = 3/2$ and $\omega_3^{1\{3\}} = 3/2$. Therefore, the strategic allocation of the initial endowments $(\omega_2^{1\{2\}}, \omega_2^{2\{2\}}) = (0, 3/2)$, $(\omega_2^{1\{2,3\}}, \omega_2^{2\{2,3\}}) = (3, 3/2)$, $(\omega_3^{1\{3\}}, \omega_3^{2\{3\}}) = (3/2, 0)$, and $(\omega_3^{1\{2,3\}}, \omega_3^{2\{2,3\}}) = (3/2, 3/2)$ maximizes simultaneously the utility functions of both player 2 and 3. Hence, this is the Nash-Walras equilibrium.

We can substitute the above results for the equilibrium price expressed by the initial endowment variables (5), and the first order conditions (3) and (4). Then, we can obtain the required results.

A.3 The latter half of Case C in Section 4.3 and Case D in Section 4.4

Let us construct the utility maximization problem for each player. For player 2,

$$\begin{array}{ll} \max_{x_2^1, x_2^2} & u_2 = \frac{8}{9} \ln x_2^1 + \frac{1}{9} \ln x_2^2 \\ \text{s.t.} & p^{1\{2,3\}} \cdot x_2^1 + 1 \cdot x_2^2 \leq p^{1\{2,3\}} \cdot 3 + 1 \cdot 3 \end{array}$$

The lagrangian of player 2's maximization problem is

$$\mathcal{L}_2 = u_2 + \lambda_2 (p^{1\{2,3\}} \cdot x_2^1 + x_2^2 - 3p^{1\{2,3\}} - 3)$$

For player 3,

$$\begin{array}{ll} \max_{x_3^1, x_3^2} & u_3 = \frac{1}{9} \ln x_3^1 + \frac{8}{9} \ln x_3^2 \\ \text{s.t.} & p^{1\{2,3\}} \cdot +x_3^1 + 1 \cdot x_3^2 \leq p^{1\{2,3\}} \cdot 3 + 1 \cdot 3 \end{array}$$

The lagrangian of player 3's maximization problem is

$$\mathcal{L}_3 = u_3 + \lambda_3 (p^{1\{2,3\}} \cdot x_3^1 + x_3^2 - 3p^{1\{2,3\}} - 3)$$

The first order conditions are the followings; $\partial \mathcal{L}_2 / \partial x_2^1 = 0$, $\partial \mathcal{L}_2 / \partial x_2^2 = 0$, $\partial \mathcal{L}_2 / \partial \lambda_2 = 0$, $\partial \mathcal{L}_3 / \partial x_3^1 = 0$, $\partial \mathcal{L}_3 / \partial x_3^2 = 0$,and $\partial \mathcal{L}_3 / \partial \lambda_3 = 0$. We can solve these equations. We have $x_2^1 = 8(p^{1\{2,3\}} + 1)/3p^{1\{2,3\}}$, $x_2^2 = p^{1\{2,3\}}/3 + 1/3$, and $\lambda_2 = -1/(3p^{1\{2,3\}} + 3)$. In addition, we have $x_3^1 = (p^{1\{2,3\}} + 1)/3p^{1\{2,3\}}$, $x_3^2 = 8p^{1\{2,3\}}/3 + 8/3$, and $\lambda_3 = -1/(3p^{1\{2,3\}} + 3)$.

The conditions of market equilibrium are $x_2^1 + x_3^1 = 3 + 3 = 6$ and $x_2^2 + x_3^2 = 3 + 3 = 6$. We can substitute x_2^1, x_3^1, x_2^2, and x_3^2 for these equations. We can solve these equations and we have the price, $p^{1\{2,3\}} = 1$. We will also substitute the solved price for the utility functions of player 2 and 3, and then, we can obtain the required results.

A.4 Case E in Section 4.5

Let us construct the utility maximization problem for each player. For player 1,

$$\begin{array}{ll} \max_{x_1^{1\{1,2\}}, x_1^{1\{1,3\}}, x_1^{2\{1,2\}}, x_1^{2\{1,3\}}} & u_1 = \frac{1}{2} \ln(x_1^{1\{1,2\}} + x_1^{1\{1,3\}}) + \frac{1}{2} \ln(x_1^{2\{1,2\}} + x_1^{2\{1,3\}}) \\ \text{s.t.} & p^{1\{1,2\}} \cdot x_1^{1\{1,2\}} + 1 \cdot x_1^{2\{1,2\}} \leq p^{1\{1,2\}} \cdot \omega_1^{1\{1,2\}} + 1 \cdot \omega_1^{2\{1,2\}} \\ & p^{1\{1,3\}} \cdot x_1^{1\{1,3\}} + 1 \cdot x_1^{2\{1,3\}} \leq p^{1\{1,2\}} \cdot \omega_1^{1\{1,3\}} + 1 \cdot \omega_1^{2\{1,3\}} \\ & \omega_1^{1\{1,2\}} + \omega_1^{1\{1,3\}} = 3 \\ & \omega_1^{2\{1,2\}} + \omega_1^{2\{1,3\}} = 3 \end{array}$$

The budget constraints of player 1 are generally kinked, that is, not differentiable in the kinked points. Therefore, we need an approach other than the Lagrangian method.

Indeed, if the budget set is not kinked, the prices should always be $p^{1\{1,2\}} = p^{1\{1,3\}}$. However, Case E is different from Case D in the point that player 2 cannot trade directly with player 3. If two prices are necessarily common, player 1 cannot coordinate excess demand or supply of both player 2 and player 3 with their different utility functions. Therefore, without loss of generality, we will assume that the equilibrium prices of two markets are different, not the same.

Assume that at the point with maximizing the utility of player 1 the gradient of tangent vector with contour of player 1's utility is intermediate between the price vector $(p^{1\{1,2\}}, 1)$ and the price vector $(p^{1\{1,3\}}, 1)$. Hence, the derivative of the implicit function $x_1^2 = f(x_1^1)$ should satisfy that

$$(6) \quad p^{1\{1,3\}} < \left| \frac{dx_1^2}{dx_1^1} \right| = \left| \frac{\partial u_1/\partial x_1^1}{\partial u_1/\partial x_1^2} \right| < p^{1\{1,2\}}$$

, which need to be confirmed after we will obtain the candidate of equilibrium.

Then, as we show the graph in Section 4.5, the point with maximizing the utility of player 1 is the kinked point which is constructed from two corners of two budget sets for the market $\{1,2\}$ and the market $\{1,3\}$. We will show again this situation as Fugure 10.

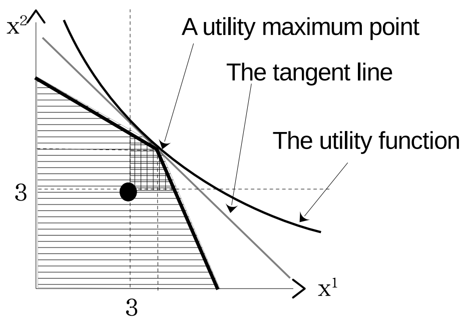


Figure 10: A budget set and a utility function in a decentralized market

Hence, each demand for each market may be the plan that one is 0 and the other is all for each good. That is, $x_1^{1\{1,2\}} = 0$ and $x_1^{2\{1,3\}} = 0$. We will substitute these conditions for the budget constraints of player 1. We will solve these equations for $x_1^{2\{1,2\}}$ and $x_1^{1\{1,3\}}$. And we can obtain

$$(7) \quad x_1^{2\{1,2\}} = -\omega_1^{1\{1,3\}} \cdot p^{1\{1,2\}} - \omega_1^{2\{1,3\}} + 3 \cdot p^{1\{1,2\}} + 3$$

and

$$(8) \quad x_1^{1\{1,3\}} = \omega_1^{1\{1,3\}} + \omega_1^{2\{1,3\}} / p^{1\{1,3\}}$$

On the other hand, for player 2 and player 3, it is sufficient to consider the ordinary utility maximization problem. Therefore, we will calculate as the same as in Case D.

For player 2,

$$\begin{array}{cl} \max_{x_2^1, x_2^2} & u_2 = \frac{8}{9} \ln x_2^1 + \frac{1}{9} \ln x_2^2 \\ \text{s.t.} & p^{1\{1,2\}} \cdot x_2^1 + 1 \cdot x_2^2 \leq p^{1\{1,2\}} \cdot 3 + 1 \cdot 3 \end{array}$$

The lagrangian of player 2's maximization problem is

$$\mathcal{L}_2 = u_2 + \lambda_2 (p^{1\{1,2\}} \cdot x_2^1 + x_2^2 - 3p^{1\{1,2\}} - 3)$$

For player 3,

$$\begin{array}{ll} \max_{x_3^1, x_3^2} & u_3 = \frac{1}{9}\ln x_3^1 + \frac{8}{9}\ln x_3^2 \\ \text{s.t.} & p^{1\{1,3\}} \cdot + x_3^1 + 1 \cdot x_3^2 \leq p^{1\{1,3\}} \cdot 3 + 1 \cdot 3 \end{array}$$

The lagrangian of player 3's maximization problem is

$$\mathcal{L}_3 = u_3 + \lambda_3(p^{1\{1,3\}} \cdot x_3^1 + x_3^2 - 3p^{1\{1,3\}} - 3)$$

The first order conditions are the followings; $\partial\mathcal{L}_2/\partial x_2^1 = 0$, $\partial\mathcal{L}_2/\partial x_2^2 = 0$, $\partial\mathcal{L}_2/\partial\lambda_2 = 0$, $\partial\mathcal{L}_3/\partial x_3^1 = 0$, $\partial\mathcal{L}_3/\partial x_3^2 = 0$,and $\partial\mathcal{L}_3/\partial\lambda_3 = 0$. We can solve these equations. We have

(9) $\quad x_2^1 = 8(p^{1\{1,2\}} + 1)/3p^{1\{1,2\}}, x_2^2 = p^{1\{1,2\}}/3 + 1/3, \textit{and } \lambda_2 = -1/(3p^{1\{1,2\}} + 3).$

In addition, we also have the followings;

(10) $\quad x_3^1 = (p^{1\{1,3\}} + 1)/3p^{1\{1,3\}}, x_3^2 = 8p^{1\{1,3\}}/3 + 8/3, \textit{and } \lambda_3 = -1/(3p^{1\{1,3\}} + 3).$

The conditions of market equilibrium are as follows. For market $\{1,2\}$, $x_1^{1\{1,2\}} + x_2^1 = \omega_1^{1\{1,2\}} + 3$ and $x_1^{2\{1,2\}} + x_2^2 = \omega_1^{2\{1,2\}} + 3$. For market $\{1,3\}$, $x_1^{1\{1,3\}} + x_3^1 = \omega_1^{1\{1,3\}} + 3$ and $x_1^{2\{1,3\}} + x_3^2 = \omega_1^{2\{1,3\}} + 3$. Previously, we assumed that $x_1^{2\{1,2\}} = 0$ and $x_1^{1\{1,3\}} = 0$. We will substitute x_2^1, x_2^2 from (9) and x_3^1, x_3^2 from (10) for these equilibrium conditions of the market. And we will solve it by prices $p^{1\{1,2\}}$ and $p^{1\{1,3\}}$ Then, we obtain

(11) $\quad p^{1\{1,2\}} = -\dfrac{8}{\omega_1^{1\{1,3\}} - 10}$

(12) $\quad p^{1\{1,2\}} = -3\omega_1^{1\{1,3\}} - 3\omega_1^{2\{1,3\}} - \dfrac{3\omega_1^{2\{1,3\}}}{p^{1\{1,3\}}} + 17$

(13) $\quad p^{1\{1,3\}} = \dfrac{1}{3\omega_1^{1\{1,3\}} \cdot p^{1\{1,2\}} + 3\omega_1^{1\{1,3\}} + 3\omega_1^{2\{1,3\}} - p^{1\{1,2\}} - 1}$

(14) $\quad p^{1\{1,3\}} = \dfrac{3\omega_1^{2\{1,3\}}}{9} + \dfrac{1}{8}$

After we obtain the required solutions, we will confirm the coincidence between (11) and (12), and between (13) and (14). Because (11) and (14) do not include the price variables, substitute (11) and (14) for the utility function of player 1 u_1. Then

$$u_1(\omega_1^{1\{1,3\}}, \omega_1^{2\{1,3\}}) = \frac{\ln\left(\omega_1^{1\{1,3\}} + \dfrac{\omega_1^{2\{1,3\}}}{\dfrac{3\omega_1^{2\{1,3\}}}{8} + \dfrac{1}{8}}\right)}{2} + \frac{\ln\left(\dfrac{8\omega_1^{1\{1,3\}}}{3\omega_1^{1\{1,3\}} - 10} - \omega_1^{2\{1,3\}} + 3 - \dfrac{24}{3\omega_1^{1\{1,3\}} - 10}\right)}{2}$$

For this utility function represented by initial endowment allocations, we will find the maximum point.

$$\frac{\partial u_1(\omega_1^{1\{1,3\}},\omega_1^{2\{1,3\}})}{\partial \omega_1^{1\{1,3\}}} = \frac{-\frac{24\omega_1^{1\{1,3\}}}{\left(3\omega_1^{1\{1,3\}}-10\right)^2}+\frac{8}{3\omega_1^{1\{1,3\}}-10}+\frac{72}{\left(3\omega_1^{1\{1,3\}}-10\right)^2}}{2\left(\frac{8\omega_1^{1\{1,3\}}}{3\omega_1^{1\{1,3\}}-10}-\omega_1^{2\{1,3\}}+3-\frac{24}{3\omega_1^{1\{1,3\}}-10}\right)} + \frac{1}{2\left(\omega_1^{1\{1,3\}}+\frac{\omega_1^{2\{1,3\}}}{\frac{3\omega_1^{2\{1,3\}}}{8}+\frac{1}{8}}\right)} = 0$$

$$\frac{\partial u_1(\omega_1^{1\{1,3\}},\omega_1^{2\{1,3\}})}{\partial \omega_1^{2\{1,3\}}} = -\frac{1}{2\left(\frac{8\omega_1^{1\{1,3\}}}{3\omega_1^{1\{1,3\}}-10}-\omega_1^{2\{1,3\}}+3-\frac{24}{3\omega_1^{1\{1,3\}}-10}\right)} + \frac{-\frac{3\omega_1^{2\{1,3\}}}{8\left(\frac{3\omega_1^{2\{1,3\}}}{8}+\frac{1}{8}\right)^2}+\frac{1}{\frac{3\omega_1^{2\{1,3\}}}{8}+\frac{1}{8}}}{2\left(\omega_1^{1\{1,3\}}+\frac{\omega_1^{2\{1,3\}}}{\frac{3\omega_1^{2\{1,3\}}}{8}+\frac{1}{8}}\right)} = 0$$

We will solve these equations and obtain four pairs of solutions; $(\omega_1^{1\{1,3\}*},\omega_1^{2\{1,3\}*}) = \left(\frac{10}{3}-\frac{2\sqrt{2}}{3},\ -\frac{1}{3}+\frac{2\sqrt{2}}{3}\right)$, $\left(\frac{10}{3}-\frac{2\sqrt{2}}{3},\ -\frac{2\sqrt{2}}{3}-\frac{1}{3}\right)$, $\left(\frac{2\sqrt{2}}{3}+\frac{10}{3},\ -\frac{1}{3}+\frac{2\sqrt{2}}{3}\right)$, $\left(\frac{2\sqrt{2}}{3}+\frac{10}{3},\ -\frac{2\sqrt{2}}{3}-\frac{1}{3}\right)$. However, the solution other than the first pair is outside the range of initial endowment. Therefore, $(\omega_1^{1\{1,3\}*},\omega_1^{2\{1,3\}*}) = \left(\frac{10}{3}-\frac{2\sqrt{2}}{3},\ -\frac{1}{3}+\frac{2\sqrt{2}}{3}\right)$ is a candidate for the Nash-Walras equilibrium of this economy. We can obtain the equilibrium price by substituting the result for (11) and (14), that is, $(p^{1\{1,2\}},p^{2\{1,2\}}) = (2\sqrt{2},1)$ and $(p^{1\{1,3\}},p^{2\{1,3\}}) = (\frac{1}{2\sqrt{2}},1)$. And also, we can confirm the coincidence between (11) and (12) and between (13) and (14) .

Lastly, we should verify the solution that satisfies the condition (6). The consumption of player 1 in the equilibrium is $(x_1^{1*},x_1^{2*}) = (6-\frac{4\sqrt{2}}{3}, 6-\frac{4\sqrt{2}}{3})$. Hence, we have

$$p^{1\{1,3\}} = \frac{1}{2\sqrt{2}} \fallingdotseq 0.3536 < \left|\frac{\partial u_1/\partial x_1^1}{\partial u_1/\partial x_1^2}\right| = 1 < p^{1\{1,2\}} = 2\sqrt{2} \fallingdotseq 2.828.$$

The equilibrium plans of player 2 and player 3 can be obtained as usual. Therefore, the results above are the equilibrium.

B Source codes for SymPy

B.1 Calculation code for Section 4.2

We will not directly solve the equilibrium in the last part below. We can confirm that the graph for utility functions substituted by required parameters is bipolar by expanding the domain of the graph, from (0,3) to (-100,100). Maybe, for this reason, we cannot directly solve the equilibrium. The calculation of the next subsection is the same process.

```
# Self-consumption vs. bilateral trade case
from sympy import Rational, symbols, diff, solve,
init_printing, ln, simplify
from sympy.plotting import plot
#initial settings
x21, x22, x31, x32 = symbols('x21, x22, x31, x32')  #demands
p1= symbols('p1') #price fo good 1
e21, e22, e31, e32
= symbols('e21, e22, e31, e32') #initial endowments
l2, l3 = symbols('l2, l3') #lagrange mulitipliers
# Utility functions
u2 = Rational(8, 9) * ln(x21) + Rational(1, 9) * ln(x22)
u3 = Rational(1, 9) * ln(x31) + Rational(8, 9) * ln(x32)
# Budget constrains
bc2 = p1 * x21 + x22 - p1 * 3 - 3
bc3 = p1 * x31 + x32 - p1 * (3-e31) - (3-e32)
# lagrangian
L2 = u2 + l2 * bc2;L3 = u3 + l3 * bc3
# Diffenciations
dx21 = diff(L2, x21);dx22 = diff(L2, x22)
dx31 = diff(L3, x31);dx32 = diff(L3, x32)
# Solve for x21, x22, and l2
solution2 = solve([dx21, dx22, bc2], [x21, x22,l2])
display(solution2)
# Solve for x31, x32, and l3
solution3 = solve([dx31, dx32,bc3], [x31, x32,l3])
display(solution3)
# Market equilibirum
mrkt_eq1 = x21 + x31 -3-(3-e31)
mrkt_eq2 = x22 + x32 -3-(3-e32)
# Substitute and solve for price
price1=solve(mrkt_eq2.subs([(x22,solution2[0][1]),
(x32,solution3[0][1])]),p1)
price2=solve(mrkt_eq1.subs([(x21,solution2[0][0]),
(x31,solution3[0][0])]),p1)
display(price1);display(price2)
# Substitute demand and initial endowment
u2_max=u2.subs([(x21,solution2[0][0]),
(x22,solution2[0][1]),(p1,price1[0])])
u3_max=u3.subs([(x31,solution3[0][0]+e31),
(x32,solution3[0][1]+e32),(p1,price1[0])])
# Differenciation by strategies(initial endowment)
sol31=diff(u3_max,e31);sol32=diff(u3_max,e32)
display(u2_max);display(u3_max);display(sol31);display(sol32)
# Search the focal point by plotting the graphs
u31plot=u3_max.subs(e32,Rational(3,2))
display(u31plot);plot(u31plot,(e31,0,3))
```

```
u32plot=u3_max.subs(e31,Rational(3,2))
display(u32plot);plot(u32plot,(e32,0,3))
# Substitute the focal point and confirm the focal point
solA=solve([sol31.subs(e31,Rational(3,2)),
sol32.subs(e31,Rational(3,2))],e32)
display(solA)
# Substitute demand and initial endowment
display(u2_max.subs(e32,Rational(3,2))
.subs(e31,Rational(3,2)))
display(u3_max.subs(e32,Rational(3,2))
.subs(e31,Rational(3,2)))
display(u2_max.subs(e32,Rational(3,2))
.subs(e31,Rational(3,2)).evalf())
display(u3_max.subs(e32,Rational(3,2))
.subs(e31,Rational(3,2)).evalf())
```

B.2 Calculation code for Section 4.3

```
# Bilateral trade with strategic
# initial endowment allocations case
from sympy import Rational, symbols, diff, solve,
init_printing, ln, simplify
from sympy.plotting import plot, plot3d
# Initial settings
x21, x22, x31, x32 = symbols('x21, x22, x31, x32') # Demands
p1= symbols('p1') # Price fo good 1
e21, e22, e31, e32
= symbols('e21, e22, e31, e32') # Initial endowments
l2, l3 = symbols('l2, l3') # lagrange mulitiplier
# Utility functions
u2 = Rational(8, 9) * ln(x21) + Rational(1, 9) * ln(x22)
u3 = Rational(1, 9) * ln(x31) + Rational(8, 9) * ln(x32)
# Budget constrains
bc2 = p1 * x21 + x22 - p1 * (3-e21) - (3-e22)
bc3 = p1 * x31 + x32 - p1 * (3-e31) - (3-e32)
# lagrangian
L2 = u2 + l2 * bc2; L3 = u3 + l3 * bc3
# Diffenciations
dx21 = diff(L2, x21); dx22 = diff(L2, x22)
dx31 = diff(L3, x31); dx32 = diff(L3, x32)
# Solve for x21, x22, and l2
solution2 = solve([dx21, dx22, bc2], [x21, x22,l2])
display(solution2)
# Solve for x31, x32, and l3
solution3 = solve([dx31, dx32,bc3], [x31, x32,l3])
display(solution3)
# Market equilibirum
mrkt_eq1 = x21 + x31 -(3-e21)-(3-e31)
mrkt_eq2 = x22 + x32 -(3-e22)-(3-e32)
# Substitute and solve for price
```

```
price1=solve(mrkt_eq2.subs([(x22,solution2[0][1]),
(x32,solution3[0][1])]),p1)
price2=solve(mrkt_eq1.subs([(x21,solution2[0][0]),
(x31,solution3[0][0])]),p1)
display(price1);display(price2)
# Substitute demand and initial endowment
u2_max=u2.subs([(x21,solution2[0][0]+e21),
(x22,solution2[0][1]+e22),(p1,price1[0])])
u3_max=u3.subs([(x31,solution3[0][0]+e31),
(x32,solution3[0][1]+e32),(p1,price1[0])])
# Differenciation by strategies(initial endowment)
sol21=diff(u2_max,e21); sol22=diff(u2_max,e22)
sol31=diff(u3_max,e31); sol32=diff(u3_max,e32)
display(u2_max);display(u3_max)
display(sol21);display(sol22);display(sol31);display(sol32)
# Search the focal point by plotting the graphs
u21plot=u2_max.subs(e22,Rational(3,2)).subs(e31,Rational(3,2))
.subs(e32,0)
display(u21plot);plot(u21plot,(e21,0,3))
solve(diff(u21plot,e21),e21)
u22plot=u2_max.subs(e21,0).subs(e31,Rational(3,2)).subs(e32,0)
display(u22plot);plot(u22plot,(e22,0,3))
u31plot=u3_max.subs(e21,0).subs(e22,Rational(3,2)).subs(e32,0)
display(u31plot);plot(u31plot,(e31,0,3))
u32plot=u3_max.subs(e21,0).subs(e22,Rational(3,2))
.subs(e31,Rational(3,2))
display(u32plot);plot(u32plot,(e32,0,3))
# Substitute the focal point and confirm the focal point
sol_all=solve([sol21.subs(e21,0).subs(e32,0),
sol22.subs(e21,0).subs(e32,0),sol31.subs(e21,0).subs(e32,0),
sol32.subs(e21,0).subs(e32,0)],[e22,e31])
display(sol_all)
# Substitute demand and initial endowment
display(u2_max.subs(e21,0).subs(e32,0)
.subs(e22,sol_all[0][0]).subs(e31,sol_all[0][1]))
display(u3_max.subs(e21,0).subs(e32,0)
.subs(e22,sol_all[0][0]).subs(e31,sol_all[0][1]))
display(u2_max.subs(e21,0).subs(e32,0)
.subs(e22,sol_all[0][0]).subs(e31,sol_all[0][1]).evalf())
display(u3_max.subs(e21,0).subs(e32,0)
.subs(e22,sol_all[0][0]).subs(e31,sol_all[0][1]).evalf())
```

B.3 Calculation code for the latter half of Section 4.3 and for Section 4.4

```
# Centralized economy
from sympy import Rational, symbols, diff, solve, init_printing, ln,simplify
#initial settings
x21, x22, x31, x32 = symbols('x21, x22, x31, x32') # demand
p1= symbols('p1') # the price of good 1
```

```
l2, l3 = symbols('l2, l3') #lagrange mulitipliers
# Utility functions
u2 = Rational(8, 9) * ln(x21) + Rational(1, 9) * ln(x22)
u3 = Rational(1, 9) * ln(x31) + Rational(8, 9) * ln(x32)
# Budget constrains
bc2 = p1 * x21 + x22 - p1 * 3 - 3
bc3 = p1 * x31 + x32 - p1 * 3 - 3
# lagrangian
L2 = u2 + l2 * bc2
L3 = u3 + l3 * bc3
# Diffenciations
dx21 = diff(L2, x21)
dx22 = diff(L2, x22)
dx31 = diff(L3, x31)
dx32 = diff(L3, x32)
init_printing()
# Solve for x21, x22, and l2
solution2 = solve([dx21, dx22, bc2], [x21, x22, l2])
display(solution2)
# Solve for x31, x32, and l3
solution3 = solve([dx31, dx32, bc3], [x31, x32, l3])
display(solution3)
# Market equilibirum
mrkt_eq1 = x21 + x31 -6
mrkt_eq2 = x22 + x32 -6
# Substitute and solve for a price
sol2=solve(mrkt_eq1.subs([(x21,solution2[0][0]),(x31,solution3[0][0])]),p1)
display(sol2)
sol3=solve(mrkt_eq2.subs([(x22,solution2[0][1]),(x32,solution3[0][1])]),p1)
display(sol3)
# Substitute demands and initial endowments for utility functions
u2_max=u2.subs([(x21,solution2[0][0]),(x22,solution2[0][1]),(p1,1)])
u3_max=u3.subs([(x31,solution3[0][0]),(x32,solution3[0][1]),(p1,1)])
display(u2_max)
display(u3_max)
display(u3_max.evalf())
```

B.4 Calculation code for Section 4.5

```
# middleman
from sympy import Rational, symbols, diff, solve, init_printing, ln,simplify,plot
#Initial Settings
x111,x121,x112,x122,x21, x22, x31, x32
= symbols('x111,x121,x112,x122,x21, x22, x31, x32') #demand
p1= symbols('p1') # price of good 1 for market 1
p2= symbols('p2') # price of good 1 for market 2
l1,l2, l3 = symbols('l1,l2, l3') # lagrange multiplyers
e11,e12 = symbols('e11,e12') # initial endowment alocations of player 1
# utility functions
u1 = Rational(1, 2) * ln(x111+x112)+ Rational(1, 2) * ln(x121+x122) # player 1
```

```
u2 = Rational(8, 9) * ln(x21) + Rational(1, 9) * ln(x22) # player 2
u3 = Rational(1, 9) * ln(x31) + Rational(8, 9) * ln(x32) # player 3
# budget constrains
bc12 = p1 * x111 + x121 - p1 * (3-e11) - (3-e12) # player 1 for market 1
bc13 = p2 * x112 + x122 - p2 * e11 - e12 # player 1 for market 2
bc2 = p1 * x21 + x22 - p1 * 3 - 3 # player 2
bc3 = p2 * x31 + x32 - p2 * 3 - 3 # player 3
#L1 = u1 + l1 * bc1
# lagragian
L2 = u2 + l2 * bc2
L3 = u3 + l3 * bc3
# first order conditions
dx21 = diff(L2, x21)
dx22 = diff(L2, x22)
dx31 = diff(L3, x31)
dx32 = diff(L3, x32)
# solve for player 1 in the kinked budget constrain
solution11 = solve(bc12.subs(x111,0),x121)
display(solution11)
solution12 = solve(bc13.subs(x122,0),x112)
display(solution12)
# Solve for x21, x22, and l2
solution2 = solve([dx21, dx22, bc2], [x21, x22, l2])
display(solution2)
# Solve for x31, x32, and l3
solution3 = solve([dx31, dx32, bc3], [x31, x32, l3])
display(solution3)
# Substitute the solutions for the budget constraints
mrkt_eq11 = 0+x21-3-(3-e11)
mrkt_eq12 = solution12[0]+x22-3-(3-e12)
mrkt_eq21 = solution11[0]+x31-3-e11
mrkt_eq22 = 0+x32-3-e12
# Solve market equilibirum
sol11=solve(mrkt_eq11.subs(x21,solution2[0][0]),p1)
display(sol11)
sol12=solve(mrkt_eq12.subs(x22,solution2[0][1]),p1)
display(sol12)
sol21=solve(mrkt_eq21.subs(x31,solution3[0][0]),p2)
display(sol21)
sol22=solve(mrkt_eq22.subs(x32,solution3[0][1]),p2)
display(sol22)
# utility functions satisfy the first order conditions
u1_max=u1.subs([(x111,0),(x112,solution12[0]),
(x121,solution11[0]),(x122,0),(p1,sol11[0]),(p2,sol22[0])])
u2_max=u2.subs([(x21,solution2[0][0]),
(x22,solution2[0][1]),(p1,sol11[0]),(p2,sol22[0])])
u3_max=u3.subs([(x31,solution3[0][0]),
(x32,solution3[0][1]),(p1,sol11[0]),(p2,sol22[0])])
display(u1_max)
#display(u1_max.evalf())
```

```
plot(u1_max.subs(e12,1),(e11,0,3))
plot(u1_max.subs(e11,2),(e12,0,3))
# maximize utility by initial endowments and solve for player 1
dfu1e1=diff(u1_max,e11)
dfu1e2=diff(u1_max,e12)
sol1=solve([dfu1e1,dfu1e2],[e11,e12])
display(sol1[0][0].evalf())
display(sol1[0][1].evalf())
display(solve([dfu1e1,dfu1e2],[e11,e12]))
display(u1_max.subs(e11,sol1[0][0]).subs(e12,sol1[0][1]))
display(u1_max.subs(e11,sol1[0][0]).subs(e12,sol1[0][1]).evalf())
# substitute equilibirum initial endowment for utility and solve for player 2
display(u2_max)
display(u2_max.subs(e11,sol1[0][0]).subs(e12,sol1[0][1]))
display(u2_max.subs(e11,sol1[0][0]).subs(e12,sol1[0][1]).evalf())
# substitute equilibirum initial endowment for utility and solve for player 3
display(u3_max)
display(u3_max.subs(e11,sol1[0][0]).subs(e12,sol1[0][1]))
display(u3_max.subs(e11,sol1[0][0]).subs(e12,sol1[0][1]).evalf())
```

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